Nonparametric Identification and Estimation of Pure Common Value Auction Models with an Application to U.S. OCS Wildcat Auctions*

(Job Market Paper)

Ming He

University of Washington

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Abstract

Although pure common value auction models have broad applicability in empirical analysis, nonparametric identification and structural estimation remain challenging in these contexts. In this paper, we establish novel identification results for both the first-price and the second-price sealed-bid auction models in the pure common value framework. We show that the joint distribution of private signals, the seller's expected profit, and the bidders' expected surplus under any reserve price are identified in a general nonparametric class. Moreover, we establish nonparametric identification of the joint distribution of private signals in a second-price sealed-bid auction model with both common-value bidders and private-value bidders. For the pure common value auction models, we propose a semiparametric estimation method and establish consistency of the estimator. Results from a Monte Carlo experiment reveal good finite sample performance of our estimator. Finally, we employ this new approach to analyze data from U.S. OCS wildcat auctions. We show that if the U.S. government had set reserve prices optimally in these auctions using the econometric method proposed in this paper, it would have increased its revenue by 15%, or 246 million dollars.

Keywords: Pure Common Value Auction, Nonparametric Identification, Copula Function, Volterra Integral Equation.

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1 Introduction

Auctions are ubiquitous in market economies. For example, the U.S. Department of the Treasury conducts weekly auctions to sell long-term securities to finance the borrowing needs of the government; the U.S. Forest Service conducts auctions to sell timbers; the U.S. federal government conducts auctions to sell mineral rights on oil and gas on the Outer Continental Shelf (OCS) off the coasts of Texas and Louisiana. There are two frameworks in the auction theory literature: the private value framework and the common value framework (see Krishna (2010) for an excellent review). In the former framework, a private-value bidder observes her private value and bids for the object for personal use. In the latter framework, a common-value bidder observes a private signal that is a proxy for the object's unknown true common value, and bids for the object for reselling purposes.

Structural econometrics of auction data was pioneered by Paarsch (1992) and Guerre, Perrigne, and Vuong (2000) (see Paarsch and Hong (2006), Athey and Haile (2007), Hendricks and Porter (2007), and Hickman, Hubbard, and Sağlam (2012) for surveys of the literature). While econometric identification and estimation has been well developed for the private value framework,[1] it is much less developed for the common value framework. The common value framework has broad applicability in many real world auctions. Examples include the eBay auctions analyzed in Bajari and Hortacsu (2003), the U.S. OCS wildcat auctions of oil-drilling rights in Hendricks, Pinkse, and Porter (2003), and among many others. However, nonparametric identification and structural estimation remain challenging in this framework. As noted in Hickman, Hubbard, and Sağlam (2012): "work on estimation in the common value paradigm has been sparse after Paarsch (1992). Identification within the common value paradigm is considerably more difficult than under private values."

One leading case of the common value framework is the pure common value model, in which all bidders have the same ex-post utility. This model is particularly relevant when all bidders face the same market selling price for the object or the same project cost at a later date. One of the most important examples is the U.S. OCS wildcat auctions of oil-drilling rights. However, all structural estimation of this data set has been conducted in the private value framework (see Li, Perrigne, and Vuong (2000, 2003), Campo, Perrigne, and Vuong (2003)). As suggested by the results in Hendricks, Pinkse, and Porter (2003), the OCS wildcat auction data is more consistent with the pure common value model than with

[1] See Guerre, Perrigne, and Vuong (2000, 2009), Li, Perrigne, and Vuong (2000, 2002, 2003), Haile and Tamer (2003), Li (2005), Campo et al. (2011), Krasnokutskaya (2011), Komarova (2011), Hubbard, Li, and Paarsch (2012), Marmer and Shneyerov (2012), Aradilla-López, Gandhi, and Quint (2013), Armstrong (2013), and Gentry and Li (2014).

the private value model. The goal of this paper is to develop econometric tools to analyze the OCS wildcat auction data set, and more generally, any data set that falls into the pure common value framework.

The nonparametric identification problem in the pure common value model is challenging for two reasons. First, the dimension of the model primitive is greater than the dimension of observed bids. In the structural auction literature, the model primitive refers to the latent joint distribution of private values in the private value framework, and it refers to the latent full joint distribution of the common value and private signals in the pure common value framework. In the latter framework, the latent full joint distribution is of one dimension greater than the observed joint distribution of bids, thus recovering it is in general impossible. Second, the standard transformation approach in the private value framework encounters a problem in the pure common value framework. Nonparametric identification in the private value framework relies on transforming the original first-order condition for the equilibrium bidding function into an equivalent form that only involves the observed distribution of bids. In this way, pseudo private values can be obtained to estimate the joint distribution of private values. In the pure common value model, however, the transformed first-order condition still involves an unknown function, which is the expectation of the common value conditional on a bidder's own signal and the highest signal among other bidders.[2] This unknown function prevents one from identifying the joint distribution of private signals, and identification of the latent full joint distribution remains even more challenging.

In this paper, instead of targeting the latent full joint distribution, we focus on policy parameters such as the seller's expected profit, the bidders' expected surplus, and the expected total welfare defined as the sum of the two. In practice, these policy parameters are more important than the latent distribution, since they can directly lead to policy analysis. Although these policy parameters can be expressed as functionals of the latent full joint distribution, we show that knowing the latent full joint distribution is sufficient but not necessary for their nonparametric identification.

We make three contributions to the structural auction literature. First, we contribute to the literature by providing novel identification results in the pure common value auction models. We analyze the exact dependence of policy parameters on the observed distributions and the unknown conditional expectation function, and establish nonparametric identifica-

[2] Other common-value bidders' private information can reveal extra information on the true common value. Wilson (1977) showed that a rational common-value bidder will take into account this information update and shade her bid to avoid the winner's curse. As a result, a common value bidder forms an expectation of the true common value conditional on her own signal and the highest signal among other bidders when maximizing her expected profit.

tion of this unknown function as well as policy parameters. Specifically, we show that for a general nonparametric class, the unknown conditional expectation function depends on the joint distribution function of private signals. By Sklar's Theorem, this joint distribution function is decomposed into the copula function and marginal distribution function of private signals. We identify the copula function of private signals from that of observed bids, and identify the inverse of the marginal distribution function–the quantile function by a Volterra integral equation of the second kind. For estimation, we propose a semiparametric method in which we parameterize the copula function but leave the marginal distribution function nonparametric. The finite dimensional parameter of the copula function is estimated by a pseudo maximum likelihood method. The quantile function of private signals is estimated by either a geometric series estimator or an iterative sieve estimator.

Second, we establish nonparametric identification results in the second-price sealed-bid auction model with both common-value bidders and private-value bidders proposed by Tan and Xing (2011). This model is more general than the second-price sealed-bid pure common value auction model. It is motivated by the observation that both types of bidders can be present in the same auction. For example, in the auctions for collectibles, such as art objects, stamps, and coins, some bidders bid for reselling purposes while others bid for their personal collections. In this model, the distribution function of private values is trivially identified since a private-value bidder bid her value. Moreover, we establish nonparametric identification of the joint distribution of private signals in a similar way as in the pure common value auction models.

Third, from an empirical point of view, we are the first to conduct structural estimation for the U.S. OCS wildcat auction data set in the pure common value framework, which has been perceived to be a more proper framework than the private value one. We estimate the copula function of private signals and find that the private signals are positively correlated. We estimate the seller's expected profit and the bidders' expected surplus to perform counterfactual analysis. We find that the actual reserve price is much lower than the optimal reserve price. Using our optimal reserve price can increase the government's revenue by 15.0%, which amounts to 246 million dollars for all the auctions considered in our sample. The optimal reserve price in the private value framework is found to be significantly different from that in the pure common value framework. If the private value model is used to guide the choice of optimal reserve price, the government's revenue will only increase by 6.8% upon the actual profit, leading to an loss of 134 million dollars compared to the maximized revenue that our optimal reserve price can generate.

This paper is related to a few papers in the literature. Paarsch (1992) imposed parametric assumptions on the private signal distribution to obtain tractable equilibrium bidding function, and used either the maximum likelihood method or the moment method to estimate the finite dimensional parameter. Li, Perrigne, and Vuong (2000) assumed the log of the unknown conditional expectation function to be of log-linear form and achieved identification up to location and scale parameters. Hendricks, Pinkse, and Porter (2003) focused on testing the rational and equilibrium bidding assumption. Février (2008) assumed a specific form of the private signal density conditional on the common value and established nonparametric identification in a particular class. Tang (2011) established bounds on the revenue distribution under counterfactual auction format and reserve price by assuming each bidder's value to be degenerate conditional on other private signals but with an unknown link function. In a general interdependent cost model, Somaini (2015) exploits exclusion restriction on the covariates (cost shifters) and achieves identification of both the joint distribution of private signals and the full information expected completion cost conditional on covariates.[3] Our paper is different from these existing works in the following ways. First, we do not assume parametric form of the unknown conditional expectation function as in Li, Perrigne, and Vuong (2000), nor do we impose the identity bidding function assumption as in Février (2008) and Tang (2011) to circumvent the problem caused by the unknown conditional expectation function. Instead, we use the data to identify this function in a nonparametric way and use this result to establish nonparametric identification of the policy parameters. Second, we do not need covariates and exclusion restrictions as in Somaini (2015). As a result, our approach can deal with any data set that falls into the pure common value framework, provided that our identification assumption is plausible for that data set. We discuss the differences between our approach and these existing approaches in detail in Section 2.2.

The rest of the paper is organized as follows. In Section 2, we review the first-price sealed-bid pure common value auction model and analyze its identification challenges. We argue that nonparametric identification of the full joint distribution of the common value and private signals is sufficient but not necessary for identification of the policy parameters. In Section 3, we show nonparametric identification of the policy parameters in both the first-price and the second-price sealed-bid auction models. We also extend our approach to establish identification of the joint distribution of private signals in a second-price sealed-bid

[3] In addition, in a procurement setting, Hong and Shum (2002) imposed parametric assumption on the joint distribution of private signals and costs, where each bidders' cost consists of a private value component and a common value component. They focused on empirically evaluating the competition effects and winner's curse effects as the number of bidders increases. Haile, Hong, and Shum (2006) developed nonparametric tests to differentiate between the private value and the common value frameworks in the first-price sealed-bid auction.

auction model with both common-value bidders and private-value bidders. In Section 4, we propose semiparametric estimators for the pure common value auction models and establish their consistency. A Monte Carlo experiment is conducted in Section 5. In Section 6, we analyze the U.S. OCS wildcat auction data set and conduct counterfactual policy analysis. Section 7 concludes. All proofs are relegated to the Appendix.

2 The Model, Identification Challenge and Policy Parameters

2.1 The Model

One indivisible good is auctioned by a first-price sealed-bid auction. Let the common value of the good be X_o with distribution function $F_{X_o}(x)$ and Lebesgue density function $f_{X_o}(x)$ on $[0, \overline{x}]$. There are M risk-neutral common-value bidders seeking to maximize their expected profits. Let the vector of private signals be $\underline{X} = (X_1, \cdots, X_M)$, distributed according to $F_{\underline{X}}(\cdot)$ with Lebesgue density function $f_{\underline{X}}(\cdot)$ on $[0, \overline{x}]^M$. The full vector $(X_o, \underline{X})$ is distributed according to $F_{X_o\underline{X}}(\cdot)$ with Lebesgue density function $f_{X_o\underline{X}}(\cdot)$ on $[0, \overline{x}]^{M+1}$. The common-value bidders are symmetric in the sense that the joint distribution function $F_{X_o\underline{X}}(\cdot)$ is invariant to any permutation of its last M arguments. The variables within $(X_o, \underline{X})$ are assumed to be affiliated as in Milgrom and Weber (1982). Let all cdfs and conditional cdfs be denoted by upper-case letters and all corresponding pdfs and conditional pdfs be denoted by lower-case letters. Specifically, for two latent random variables X, Y, we use $F_{XY}(x, y)$ to denote the joint cdf, use $F_{X|Y}(x|y), f_{X|Y}(x|y)$ to denote the conditional cdf and pdf, respectively, and use $F_X(x), f_X(x)$ to denote the marginal cdf and pdf, respectively. For two observed random variables, we use G, g to replace F, f, respectively in the above notations.

We focus on bidder 1 due to symmetry among the bidders. In the pure common value framework, bidder 1 does not observe the realization of X_o prior to the auction, but observes her private signal $X_1 = x$. Given that other bidders follow the same equilibrium bidding strategy $\beta(\cdot)$, bidder 1 chooses a bid b to maximize her expected profit

$$\pi(b; x) = \mathrm{E}[(X_o - b)\mathbb{1}(\beta(Y_1) \leq b)|X_1 = x], \tag{2.1}$$

where $Y_1 = \max X_{-1}$, $X_{-1} = (X_2, \cdots, X_M)$, and $\mathbb{1}(\cdot)$ is the indicator function. The first-order condition and definition of Bayesian Nash equilibrium lead us to the equilibrium bidding function that satisfies the differential equation

$$\beta'(x) = \Big[\overline{H}(x) - \beta(x)\Big]\rho_{Y_1|X_1}(x), \tag{2.2}$$

subject to the boundary condition $\beta(0) = \overline{H}(0)$, where $\rho_{Y_1|X_1}(x) = f_{Y_1|X_1}(x|x)/F_{Y_1|X_1}(x|x)$ is the reverse hazard function of Y_1 conditional on X_1 evaluated at the diagonal and $\overline{H}(x) =$

$\mathrm{E}[X_o|X_1 = x, Y_1 = x]$. The equilibrium bidding function can be solved as

$$\beta(x) = \overline{H}(x) - \int_0^x J(a|x) d\overline{H}(a),\ x \in [0, \overline{x}], \tag{2.3}$$

where $J(a|x) = \exp\left(-\int_a^x \rho_{Y_1|X_1}(s)ds\right)$. The function $\overline{H}(x)$ represents bidder 1's expectation of the common value conditional on her signal and on her equilibrium bid being pivotal. Hong, Haile, and Shum (2006) term it the conditional expected valuation. In the analysis below, we show that it plays a key role for the nonidentification result in the pure common value auction model.

2.2 Identification Challenges and Existing Approaches

Given a random sample of bids $\{B_{1\ell}, \cdots, B_{M\ell}\}_{\ell=1}^L$ from L auctions with non-binding reserve price, we can let $B_1 = \beta(X_1)$ and $M_1 = \beta(Y_1)$, where M_1 is the maximum bid from bidder 1's competitors. Let $G_{M_1|B_1}(m_1|b_1)$ and $g_{M_1|B_1}(m_1|b_1)$ be the distribution function and density function of M_1 conditional on B_1, and $\rho_{M_1|B_1}(b) = g_{M_1|B_1}(b|b)/G_{M_1|B_1}(b|b)$ be the reverse hazard function of M_1 conditional on B_1 evaluated at the diagonal. Applying the standard GPV type transformation (see Guerre, Perrigne, and Vuong (2000), Li, Perrigne, and Vuong (2002)), it can be easily shown that $\rho_{M_1|B_1}(\beta(x)) = \rho_{Y_1|X_1}(x)/\beta'(x)$, and we can write (2.2) as

$$x = \overline{H}^{-1}\left(b + \frac{1}{\rho_{M_1|B_1}(b)}\right), \tag{2.4}$$

where $b = \beta(x)$ and $\overline{H}^{-1}(x)$ is the inverse function of $\overline{H}(x)$.

The standard GPV type transformation for nonparametric identification encounters two challenges in the pure common value auction model. First, even if the private signals could be estimated from the observed bids by (2.4), using an M dimensional pseudo signals to recover the $M+1$ dimensional joint distribution of the common value and the private signals is not possible. Second, the key component, the conditional expected valuation function, is unknown. This prevents us from obtaining pseudo signals from the observed bids and bids distributions as represented by the reverse hazard function of M_1 conditional on B_1 above.

Several attempts have been made to deal with the above two challenges. First, it is common to adopt the mineral rights model, which is a special case of the pure common value model. In this model, the private signals are assumed to be i.i.d. conditional on the common value X_o. In this case, the joint distribution of the common value and the private signals is reduced to the marginal density function $f_{X_o}(x)$ and conditional density function $f_{X_1|X_o}(x_1|x_o)$. Previous works under this framework include Paarsch (1992), Li, Perrigne, and Vuong (2000), and Février (2008). For identification in the mineral rights model, more assumptions are needed. Paarsch (1992) parameterized the marginal and conditional density functions to special families in order to yield a tractable equilibrium bidding function and used either the maximum likelihood method or the moment method to estimate the finite dimensional parameters. Li, Perrigne, and Vuong (2000) assumed a multiplicative decomposition of the form $X_m = X_o\epsilon_m$ with $X_o \perp \epsilon_m$ for i.i.d. $\epsilon_1, \cdots, \epsilon_M$, where "$\perp$" denotes the statistical independence. Their model was defined by $f_{X_o}(\cdot)$ and the density function $f_\epsilon(\cdot)$ of the idiosyncratic term. Février (2008) assumed a very specific nonparametric structure of the conditional density function.

Second, the unknown conditional expected valuation did not pose a problem in Paarsch (1992) since the inverse transformation was not needed in his parametric approach. Li, Perrigne, and Vuong (2000) assumed this function to be of the form $\overline{H}(x) = a_1 x^{a_2}$ for some constants a_1 and a_2, which restricted the full joint distribution of the common value and the private signals to an unknown space of functions. Février (2008) and Tang (2011) both adopted a normalization assumption that the equilibrium bidding function is the identity function to circumvent the problem, and this also restricted the full joint distribution to an unknown space of functions. For the identification results, Paarsch (1992) achieved identification under parameterization; Li, Perrigne, and Vuong (2000) identified the functions $f_{X_o}(\cdot)$ and $f_\epsilon(\cdot)$ up to the parameters a_1, a_2 using the Kotlarski decomposition; Février (2008) achieved nonparametric identification in a particular class; and Tang (2011) employed a partial identification approach and focused on bounding the seller's expected profit under counterfactual auction format and reserve price.

In this paper, we take a step back and pose two questions: First, instead of making the identity bidding function normalization or assuming certain forms to deal with the unknown conditional expected valuation function, can we identify it from the data under some weak assumptions on the full joint distribution of the common value and the private signals? Second, although the full joint distribution is sufficient for identifying any functional of the model primitive, for particular functionals of interest such as the seller's expected profit and bidders' expected surplus under any reserve price, is it necessary to identify the full joint distribution? We address the two questions in the following sections.

2.3 The Seller's Expected Profit and Bidders' Expected Surplus Under Counterfactual Reserve Price

Let $\overline{L}(x) = \mathrm{E}[X_o|X_1 = x, Y_1 \leq x]$. From Milgrom and Weber (1982), the equilibrium bidding function in a first-price sealed-bid pure common value auction under reserve price

$r \in [\overline{L}(0), \overline{L}(\overline{x})]$ is

$$\beta_r(x) = rJ(x_r^*|x) + \int_{x_r^*}^{x} \overline{H}(a)dJ(a|x), \; x \in [x_r^*, \overline{x}],$$

where $x_r^* = \inf_{x \in [0,\overline{x}]}\{\overline{L}(x) \geq r\}$ with $\beta_r(x_r^*) = r$. In the rest of this paper, we call $\overline{H}(x)$ and $\overline{L}(x)$ the high and low conditional expected valuations, respectively. Policy makers are often interested in how the seller's expected profit and bidders' expected surplus change with reserve price r since these policy parameters depend on $\beta_r(x)$. We illustrate the welfare implications of reserve price in the following example.

Example 2.1 Let the private signals $\{X_m\}_{m=1}^M$ be i.i.d. uniformly distributed on $(0, 1)$, $X_o = \sum_{m=1}^M X_m/M + \epsilon$, where $M = 3$, $\epsilon \perp \underline{X}$ with $\mathrm{E}[\epsilon] = 0$. Let the seller's own valuation v_o be 0.25. In a first-price sealed-bid auction in the pure common value framework, $\beta_r(x) = \frac{3r^3}{8x^2} + \frac{5x}{9}$ for $(x, r) \in [\frac{3r}{2}, 1] \times [0, \frac{2}{3}]$. It can be shown that

$$\mathrm{E}[\pi_S(r)] = \begin{cases} -\frac{243}{64}r^4 + \frac{63}{32}r^3 + \frac{5}{12} & r \in [0, \frac{2}{3}] \\ \frac{1}{4} & r \in (\frac{2}{3}, 1] \end{cases}, \; \mathrm{E}[\pi_B(r)] = \begin{cases} \frac{81}{64}r^4 - \frac{9}{8}r^3 + \frac{1}{12} & r \in [0, \frac{2}{3}] \\ 0 & r \in (\frac{2}{3}, 1] \end{cases},$$

where $\mathrm{E}[\pi_S(r)]$ and $\mathrm{E}[\pi_B(r)]$ are the seller's expected profit and bidders' expected surplus, respectively. $\mathrm{E}[\pi_S(r)]$ is maximized at $r = 0.39$ with value 0.446. The expected total welfare, defined as $\mathrm{E}[\pi_S(r)] + \mathrm{E}[\pi_B(r)]$, is maximized at $r = v_o = 0.25$ with value 0.503. If r increases from 0.25 to 0.39, there will be a 3.3% increase in the seller's expected profit, accompanied by a 1.9% loss in the expected total welfare.

In addition, we emphasize the important implications of model specification on the policy parameters. To illustrate, we compare the two policy parameters in the pure common value framework and in the private value framework. It can be shown that in the private value framework, $\beta_r(x) = \frac{2}{3}x + \frac{r^3}{3x^2}$ for $x \in [r, 1]$, and

$$\mathrm{E}[\pi_S(r)] = -\frac{3}{2}r^4 + \frac{5}{4}r^3 + \frac{1}{2}, \; \mathrm{E}[\pi_B(r)] = \frac{3}{4}r^4 - r^3 + \frac{1}{4}, r \in [0, 1].$$

Different curves are plotted in Figure 1. If the true framework is the pure common value model but incorrectly specified as a private value one, the seller's expected profit will move from point A to B with a 27% loss.

In general, we can follow a similar idea as in Li, Perrigne, and Vuong (2003) to write the above two policy parameters in terms of observed quantities as shown in the following proposition.

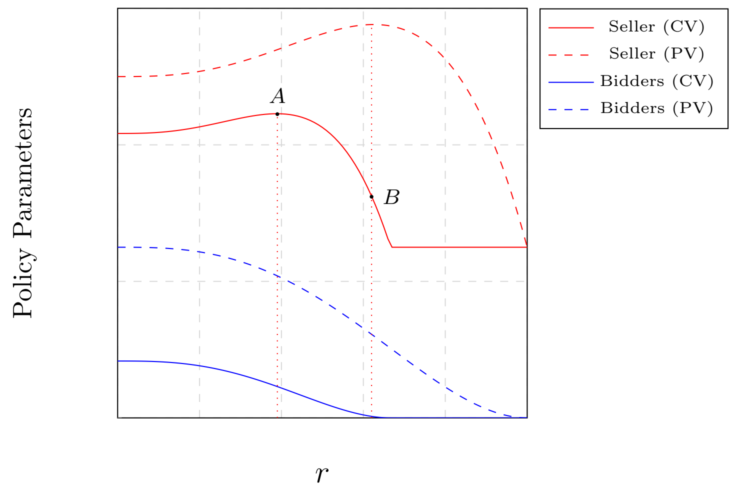


Figure 1: Reserve Price and Policy Parameters under Different Frameworks

Proposition 2.2 Let v_o be the seller's own valuation of the object. In a first-price sealed-bid pure common value auction, the seller's expected profit and the bidders' expected surplus under reserve price r are

$$\begin{aligned} \mathrm{E}[\pi_S(r)] &= v_o \mathrm{E}\left[\mathbb{1}(B^{(M)} < b_r^*)\right] + \mathrm{E}[\pi_P(r)], & (2.5) \\ \mathrm{E}[\pi_B(r)] &= \mathrm{E}\left[\mathbb{1}(M_1 \geq b_r^*)\mathbb{1}(B_1 \leq M_1)\overline{L}(\beta^{-1}(M_1))\right] \\ &+ \mathrm{E}\left[\mathbb{1}(B_1 \geq b_r^*)\mathbb{1}(M_1 \leq B_1)\overline{L}(\beta^{-1}(B_1))\right] - \mathrm{E}[\pi_P(r)], & (2.6) \end{aligned}$$

where $\mathrm{E}[\pi_P(r)] = \mathrm{E}\left[\left(B^{(M)} + (r - b_r^*)J^*(b_r^*|B^{(M)})\right)\mathbb{1}(B^{(M)} \geq b_r^*)\right]$ is the expected payment from the bidders when the object is sold. $B^{(M)} = B_1 \vee M_1$ is the maximum bid, $J^*(b_r^*|B^{(M)}) = \exp\left(-\int_{b_r^*}^{B^{(M)}} \rho_{M_1|B_1}(t)dt\right)$, $b_r^* = \beta(x_r^*)$, and $\beta^{-1}(b) = \overline{H}^{-1}(b + \frac{1}{\rho_{M_1|B_1}(b)})$.

Proof. See the Appendix. ■

Given a random sample of equilibrium bids, $\mathrm{E}[\pi_S(r)]$ and $\mathrm{E}[\pi_B(r)]$ will be nonparametrically identified if $\overline{H}(x), \overline{L}(x)$, and b_r^* are known. By construction, b_r^* solves

$$\overline{H}(x_r^*) = b_r^* + \frac{1}{\rho_{M_1|B_1}(b_r^*)}. \tag{2.7}$$

The definition of x_r^* implies that solving b_r^* requires information on the functions $\overline{H}(x)$ and $\overline{L}(x)$. Therefore, for both $\mathrm{E}[\pi_S(r)]$ and $\mathrm{E}[\pi_B(r)]$, the essential unknowns are the two conditional expected valuation functions $\overline{H}(x)$ and $\overline{L}(x)$. In the next section, we show that they are nonparametrically identified under a weak assumption on the joint distribution of the common value and the private signals. This implies that both the seller's expected profit and the bidders' expected surplus under any reserve price are nonparametrically identified.

3 Nonparametric Identification

3.1 Identification in the First-Price Sealed-Bid Pure Common Value Auction

First-price sealed-bid auctions are prevalent in the real world. Examples include the U.S. OCS wildcat auctions (Hendricks, Pinkse, and Porter (2003)), the U.S. highway procurement auctions (Li and Zheng, 2009), and the competitive sales of U.S. municipal bonds (Tang, 2011). In this section, we show that in the first-price sealed-bid pure common value auction, both the seller's expected profit and the bidders' expected surplus under any reserve price are nonparametrically identified under a weak assumption on the joint distribution of the common value and the private signals.

The basic idea in our identification approach is as follows. From the analysis in Section 2.3, the essential unknowns in evaluating the two policy parameters are the two conditional expected valuation functions, which are functionals of the latent full joint distribution $F_{X_o\underline{X}}(\cdot)$. In general, X_o is unobserved thus posing difficulty for identifying the two conditional expected valuation functions. If we can reduce these two functions as functionals of the latent joint distribution $F_{\underline{X}}(\cdot)$ of private signals, then by Sklar's theorem (see Nelsen (2006)), we can write

$$F_{\underline{X}}(x_1, \cdots, x_M) = C_o(F_o(x_1), F_o(x_2), \cdots, F_o(x_M)), \tag{3.1}$$

where $C_o(\cdot)$ denotes the true copula function of private signals, and we use $F_o(\cdot)$ to denote the true marginal distribution function of X_1 to minimize notation. The copula function is unique since $F_o(x)$ is absolutely continuous. Since the observed bids are strictly increasing transformations of the private signals, it can be shown that the copula function of the private signals is the same as the copula function of the observed bids. As a result, the copula function of private signals is directly identified from the sample. The only unknown is the marginal distribution function, but the first-order condition in either (2.2) or (2.4) can be used as a restriction to reduce the space of distribution functions that it lies in. If the restricted space turns out to be a singleton, we achieve point identification. To reduce the dimension such that the two conditional expected valuation functions are functionals of $F_{\underline{X}}(\cdot)$, it is most natural to assume the following.

Assumption (AS) $X_o = \frac{1}{M}\sum_{m=1}^{M} X_m + \epsilon$, where $\epsilon \perp \underline{X}$ and $\mathrm{E}[\epsilon] = 0$.[4]

[4]Alternatively, we can assume an independent multiplicative error, that is, $X_o = (\frac{1}{M}\sum_{m=1}^{M} X_m)\epsilon$, where $\epsilon \perp \underline{X}$ and $\mathrm{E}[\epsilon] = 1$. Moreover, from our derivation in the Appendix, the simple mean function in Assumption (AS) can be extended to other known function of $\underline{X}$, but the resulting integral equation that restricts the

The idea of Assumption (AS) is that each common-value bidder has partial information on the true common value. For example, bidders in the OCS wildcat oil-drilling auction might conduct their own seismic surveys of an oil tract, bidders in an automobile auction might bring their own mechanics to learn about the true conditions of the cars. In such examples, bidders' signals are equally informative. Each bidder forms an imprecise estimate of the true common value and the future common value is likely to be the average of these partial information up to some stochastic error term. Previous papers that have used the average formulation include Klemperer (1998), Goeree and Offerman (2002, 2003), where they assume $X_o = \frac{1}{M}\sum_{m=1}^{M} X_m$. Our assumption is weaker since we do not assume a deterministic relation between X_o and $\underline{X}$. In particular, Assumption (AS) implies that the conditional mean of the common value is the simple average of private signals, that is, $\mathrm{E}[X_o|\underline{X}] = \frac{1}{M}\sum_{m=1}^{M} X_m$.

Lemma 3.1 Under Assumption (AS),

$$\overline{H}(x) \equiv \overline{H}(x; F_o, C_o) = x - \frac{M-2}{M}\frac{\int_0^x C_{o,12}(F_o(x), F_o(x), F_o(t), F_o(x), \cdots, F_o(x))dt}{C_{o,12}(F_o(x), \cdots, F_o(x))},$$
$$\overline{L}(x) \equiv \overline{L}(x; F_o, C_o) = x - \frac{M-1}{M}\frac{\int_0^x C_{o,1}(F_o(x), F_o(t), F_o(x), \cdots, F_o(x))dt}{C_{o,1}(F_o(x), \cdots, F_o(x))},$$

where $F_o(x)$ is the distribution function of X_1, $C_o(\underline{u})$, $\underline{u} = (u_1, \cdots, u_M)$, is the true copula function of the private signals, $C_{o,1}(\underline{u}) = \partial C_o(\underline{u})/\partial u_1$, and $C_{o,12}(\underline{u}) = \partial^2 C_o(\underline{u})/\partial u_1 \partial u_2$.

Proof. See the Appendix. ■

Now $\overline{H}(x)$ and $\overline{L}(x)$ are known up to the marginal distribution function $F_o(x)$. If we are willing to make an assumption that the marginal distribution function is known, and in particular, is the uniform distribution on the unit interval as in Somaini (2015), then the two conditional expected valuation functions are identified. However, we will show below that under Assumption (AS), the first-order condition in (2.4) actually imposes a restriction on the marginal distribution and we cannot arbitrarily assume a known marginal distribution.

Due to the fact that $B_1 = \beta(X_1)$, we have $\beta(x) = Q_{B_1}(F_o(x))$, where $Q_{B_1}(\cdot)$ is the quantile function of B_1. Combining this with the transformed first-order condition in (2.4), we obtain $R_1(x; F_o, C_o) = 0$, where

$$R_1(x; F, C_o) = \overline{H}(x; F, C_o) - Q_{B_1}(F(x)) - \frac{1}{\rho_{M_1|B_1}(Q_{B_1}(F(x)))}. \tag{3.2}$$

signal quantile function could be nonlinear.

The relation $R_1(x; F_o, C_o) = 0$ imposes a restriction on the distribution function $F_o(x)$.[5] We show in the following theorem that $F_o(x)$ or equivalently, the true quantile function $Q_o(\tau) = F_o^{-1}(\tau), \tau \in [0,1]$ of private signals, is nonparametrically identified by this restriction under Assumption (CU-1) below. Therefore, both $\overline{H}(x)$ and $\overline{L}(x)$ are nonparametrically identified and as a result, $\mathrm{E}[\pi_S(r)]$ and $\mathrm{E}[\pi_B(r)]$ are nonparametrically identified. Let

$$\phi_{1o}(\tau) = \frac{M}{2}\left(Q_{B_1}(\tau) + \frac{1}{\rho_{M_1|B_1}(Q_{B_1}(\tau))}\right),\ k_{1o}(\tau, s) = -\frac{(M-2)}{2} z_{1o}(\tau, s),$$

where $z_{1o}(\tau, s) = \frac{C_{o,123}(\tau,\tau,s,\tau,\cdots,\tau)}{C_{o,12}(\tau,\cdots,\tau)}$ and $C_{o,123}(\underline{u}) = \partial^3 C_o(\underline{u})/\partial u_1 \partial u_2 \partial u_3$. We further make the following assumption.

Assumption (CU-1) $\phi_{1o}(\tau)$ is continuous on $[0,1]$, $k_{1o}(\tau, s)$ is continuous on $[0,1]^2$.

Theorem 3.2 In the first-price sealed-bid pure common value auction model, under Assumptions (AS) and (CU-1), the true quantile function $Q_o(\tau)$ of private signal is nonparametrically identified as the unique solution to the following Volterra integral equation of the second kind,

$$Q(\tau) - \int_0^\tau k_{1o}(\tau, s) Q(s) ds = \phi_{1o}(\tau). \tag{3.3}$$

Proof. see the Appendix.[6] ■

Remark 3.3 In the nonparametric instrumental regression problem, a Fredholm integral equation of the first kind is typically involved. In that case, the inverse problem is ill-posed and regularized estimator is needed, see Darolles, Fan, Florens, and Renault (2011).

[5]This is a more general view on the identification of the marginal distribution function (or equivalently the quantile function). In the affiliated private value framework (which nests the independent private value case), the private value can be expressed as a closed form of observed distributions. As a result, the private value quantile function can be explicitly expressed as a closed form of the observed quantile function of bids (see Marmer and Shneyerov (2012), Fan, Li, and Pesendorfer (2015), and Fan, He, and Li (2015)). In particular, we have $x = b + \frac{1}{\rho_{M_1|B_1}(b)}$ in the affiliated private value framework, where $b = \beta(x)$. Upon the substitution $\beta(x) = Q_{B_1}(F_o(x))$ and change of variable, we get $Q_o(\tau) = Q_{B_1}(\tau) + \frac{1}{\rho_{M_1|B_1}(Q_{B_1}(\tau))}$.

[6]A special case occurs when $M = 2$. In this case, the identification of the signal quantile function reduces exactly to that in the affiliated private value framework. This is because $\overline{H}(x) = x$ when $M = 2$ in the pure common value model under Assumption (AS), which is the same as that in the private value model. This implies that the two models will generate the same equilibrium bidding function under no reserve price and thus the same observed distributions. However, this does not imply that they will generate the same equilibrium bidding functions under any reserve price and as a consequence, the seller's expected profit and the bidders' expected surplus under any reserve price will be different. In fact, a simple calculation would reveal that the two models generate very different policy parameters even when $M = 2$ in the setup of Example 2.1.

In contrast, the above Volterra integral equation of the second kind is well-posed in the sense that the inverse operator $(I - K_{1o})^{-1}$ of $I - K_{1o}$ exists and is a continuous, where $K_{1o}Q(\tau) = \int_0^\tau k_{1o}(\tau, s)Q(\tau)ds$ (see more discussion in Section 4.2). Given that $Q_o(\cdot)$ is nonparametrically identified, the joint distribution function $F_{\underline{X}}(\cdot)$ is nonparametrically identified by (3.1). As a result, the two conditional expected valuation functions are nonparametrically identified by Lemma 3.1 and the seller's expected profit and the bidders' expected surplus under any reserve price are nonparametrically identified by Proposition 2.2. In addition, our identification approach can be extended to the case when only the highest two bids are observed as in Fan, He, and Li (2015). By assuming the copula function to be in a nonparametric Archimedean class with weak requirement, the results in Theorem 3.2 of Fan, He, and Li (2015) can be used to identify the copula function, and our Theorem 3.2 above implies identification of the joint distribution of private signals and thus the two policy parameters.

Example 3.4 Consider the same setup as in Example 2.1. The equilibrium bidding strategy in a first-price sealed-bid auction is $\beta(x) = 5x/9$ for $x \in [0, 1]$. In this case, $\rho_{M_1|B_1}(b) = 2/b$ for $b \in [0, 5/9]$, $Q_{B_1}(\tau) = 5\tau/9$ for $\tau \in [0, 1]$, and $\overline{H}(x; F, C_\perp) = x - \int_0^x F(t)dt/[3F(x)]$. The Volterra integral equation becomes

$$Q(\tau) + \frac{1}{2\tau}\int_0^\tau Q(s)ds = \frac{5\tau}{4},$$

with the unique solution $Q_o(\tau) = \tau, \tau \in [0, 1]$.

3.2 Identification in Second-Price Sealed-Bid Pure Common Value Auction

We consider the second-price sealed-bid auction in this section. In the pure common value framework, it is well known that the equilibrium bidding function is $\beta(x) = \overline{H}(x)$, $x \in [0, \overline{x}]$. Under Assumption (AS), we can make use of the relation $\beta(x) = Q_{B_1}(F_o(x))$ and write

$$R_2(x; F, C_o) = \overline{H}(x; F, C_o) - Q_{B_1}(F(x)). \tag{3.4}$$

Then the function $F_o(\cdot)$ satisfies the restriction $R_2(x; F_o, C_o) = 0$. We show in the following theorem that in the second-price sealed-bid auction, the quantile function $Q_o(\cdot)$ of private signal is also nonparametrically identified by an integral equation similar to that in Theorem 3.2. Let

$$\phi_{2o}(\tau) = \frac{MQ_{B_1}(\tau)}{2},\ k_{2o}(\tau, s) = k_{1o}(\tau, s),$$

where $k_{1o}(\tau, s)$ is defined in Theorem 3.2.

Assumption (CU-2) $\phi_{2o}(\tau)$ is continuous on $[0, 1]$, $k_{2o}(\tau, s)$ is continuous on $[0, 1]^2$.

Theorem 3.5 In a second-price sealed-bid pure common value auction, under Assumptions (AS) and (CU-2), the true quantile function $Q_o(\tau)$ of private signal is nonparametrically identified as the solution to the following Volterra integral equation of the second kind,

$$Q(\tau) - \int_0^\tau k_{2o}(\tau, s)Q(s)ds = \phi_{2o}(\tau), \tag{3.5}$$

Proof. Follows a similar argument as in the proof of Theorem 3.2. ■

Example 3.6 We continue Example 3.4 but in a second-price sealed-bid auction with M bidders. The equilibrium bidding strategy is $\beta(x) = (M+2)x/(2M)$ for $x \in [0,1]$. In this case, $\rho_{M_1|B_1}(b) = (M-1)/b$ for $b \in [0,(M+2)(M-1)/(2M^2)]$, $Q_{B_1}(\tau) = (M+2)\tau/(2M)$ for $\tau \in [0,1]$, and $\overline{H}(x; F, C_\perp) = x - (M-2)\int_0^x F(t)dt/[MF(x)]$. The Volterra integral equation becomes

$$Q(\tau) + \frac{M-2}{2\tau}\int_0^\tau Q(s)ds = \frac{(M+2)\tau}{4}.$$

It is easy to verify that the unique solution is $Q_o(\tau) = \tau, \tau \in [0,1]$.

In a second-price sealed-bid auction, the equilibrium bidding strategy under reserve price $r \in [\overline{L}(0), \overline{L}(x)]$ is shown in Milgrom and Weber (1982) to be $\beta_r(x) = \overline{H}(x), x \in [x_r^*, \overline{x}]$ and $\beta_r(x) < r, x \in [0, x_r^*)$. We show in the following proposition that identification of the two conditional expected valuations are sufficient for identifying the seller's expected profit and the bidders' expected surplus under any reserve price.

Proposition 3.7 Let v_o be the seller's own valuation of the object. In a second-price sealed-bid pure common value auction, the seller's expected profit and the bidders' expected surplus under reserve price $r \in [\overline{L}(0), \overline{L}(x)]$ are

$$\begin{aligned}
\mathrm{E}[\pi_S(r)] &= v_o\mathrm{E}[\mathbb{1}(B^{(M)} < \overline{H}(x_r^*))] + \mathrm{E}[\pi_P(r)], &(3.6)\\
\mathrm{E}[\pi_B(r)] &= \mathrm{E}\left[\mathbb{1}(M_1 \geq \overline{H}(x_r^*))\mathbb{1}(B_1 \leq M_1)\overline{L}(\overline{H}^{-1}(M_1))\right] \\
&+ \mathrm{E}\left[\mathbb{1}(B_1 \geq \overline{H}(x_r^*))\mathbb{1}(M_1 \leq B_1)\overline{L}(\overline{H}^{-1}(B_1))\right] - \mathrm{E}[\pi_P(r)], &(3.7)
\end{aligned}$$

where

$$\mathrm{E}[\pi_P(r)] = r\mathrm{E}[\mathbb{1}(B^{(M)} \geq \overline{H}(x_r^*), B^{(M-1)} < \overline{H}(x_r^*))] + \mathrm{E}[B^{(M-1)}\mathbb{1}(B^{(M-1)} \geq \overline{H}(x_r^*))]$$

is the expected payment from the bidders when the object is sold, $B^{(M)}, B^{(M-1)}$ are the highest and second-highest bids, respectively.

Proof. See the Appendix. ■

3.3 Extension to the Second-Price Sealed-Bid Auction with both Common-Value Bidders and Private-Value Bidders

Tan and Xing (2011) proposed a second-price sealed-bid auction model in which both common-value bidders and private-value bidders are present. This framework is particularly relevant for auctions for collectibles, such as art objects, stamps, and coins, in which some bidders bid for reselling purposes while others bid for their personal collections. There are M symmetric common-value bidders and N symmetric private-value bidders. The common-value bidders' information is independent of the private-value bidders' information. The Bayesian Nash equilibrium (BNE) strategy profile is characterized in the following theorem.

Theorem 3.8 *(Proposition 1 (Tan and Xing (2011)))* In a second-price sealed-bid auction with both common-value bidders and private-value bidders, a monotone pure-strategy BNE exists. In equilibrium, private-value bidders bid their private values, $\beta_p(v) = v$, and common-value bidders bid according to $\beta(x)$ which satisfies the first-order differential equation

$$\beta'(x) = \frac{\rho_{Y_1|X_1}(x)[\overline{H}(x) - \beta]}{N\rho_V(\beta)[\beta - \overline{L}(x)]}, \tag{3.8}$$

subject to the boundary condition $\beta(0) = \overline{H}(0) = \overline{L}(0)$, where $\rho_V(v) = f_V(v)/F_V(v)$ is the reverse hazard function of the i.i.d. private values, and $\rho_{Y_1|X_1}(x) = f_{Y_1|X_1}(x|x)/F_{Y_1|X_1}(x|x)$ is the reverse hazard function of Y_1 conditional on $X_1 = x$.

The distribution function of the private value is trivially identified due to the private-value bidders' identity bidding function. For identification of the joint distribution of common-value bidders' private signals, we have $\rho_{Y_1|X_1}(x) = \rho_{M_1|B_1}(\beta(x))\beta'(x)$, where B_1 is bid from common-value bidder 1, and M_1 is the highest bid from other common-value bidders. The transformed first-order condition for common-value bidder 1 is

$$\rho_{M_1|B_1}(\beta(x))[\overline{H}(x) - \beta(x)] + N\rho_V(\beta(x))[\overline{L}(x) - \beta(x)] = 0. \tag{3.9}$$

Under Assumption (AS), we can make use of the relation $\beta(x) = Q_{B_1}(F_o(x))$ and write

$$\begin{aligned} R_3(x; F, C_o) = {} & \rho_{M_1|B_1}(Q_{B_1}(F(x))) \left[\overline{H}(x; F, C_o) - Q_{B_1}(F(x))\right] \\ & + N\rho_V(Q_{B_1}(F(x))) \left[\overline{L}(x; F, C_o) - Q_{B_1}(F(x))\right]. \end{aligned} \tag{3.10}$$

The common-value bidders' signal distribution function $F_o(\cdot)$ is subject to the restriction $R_3(x; F_o, C_o) = 0$. Let

$$\phi_{3o}(\tau) = \frac{M\left[\rho_{M_1|B_1}(Q_{B_1}(\tau)) + N\rho_V(Q_{B_1}(\tau))\right] Q_{B_1}(\tau)}{2\rho_{M_1|B_1}(Q_{B_1}(\tau)) + N\rho_V(Q_{B_1}(\tau))},$$

$$k_{3o}(\tau, s) = -\frac{(M-2)\rho_{M_1|B_1}(Q_{B_1}(\tau)) z_{1o}(\tau, s)}{[2\rho_{M_1|B_1}(Q_{B_1}(\tau)) + N\rho_V(Q_{B_1}(\tau))]} - \frac{N(M-1)\rho_V(Q_{B_1}(\tau)) z_{2o}(\tau, s)}{[2\rho_{M_1|B_1}(Q_{B_1}(\tau)) + N\rho_V(Q_{B_1}(\tau))]},$$

where $z_{1o}(\tau, s)$ is defined in Theorem 3.2, $z_{2o}(\tau, s) = \frac{C_{o,12}(\tau,s,\tau,\cdots,\tau)}{C_{o,1}(\tau,\cdots,\tau)}$, and $C_{o,1}(\underline{u}) = \partial C_o(\underline{u})/\partial u_1$.

Assumption (CU-3) $\phi_{2o}(\tau)$ is continuous on $[0, 1]$, $k_{2o}(\tau, s)$ is continuous on $[0, 1]^2$.

Theorem 3.9 In a second-price sealed-bid auction with both common-value bidders and private-value bidders, under Assumptions (AS) and (CU-3), the quantile function $Q_o(\tau)$ of each common-value bidder is nonparametrically identified as the solution to the following Volterra integral equation of the second kind,

$$Q(\tau) - \int_0^\tau k_{3o}(\tau, s)Q(s)ds = \phi_{3o}(\tau). \tag{3.11}$$

Proof. See the Appendix. ■

Example 3.10 Consider the case that the private values $\{V_n\}_{n=1}^N$ and the signals $\{X_m\}_{m=1}^M$ are i.i.d. uniformly distributed on $(0, 1)$, the common value satisfies Assumption (AS). Let $N = M = 3$, then the equilibrium bidding strategy pair is $(\beta(x), \beta_p(v)) = (11x/15, v)$. In this case, $\rho_{M_1|B_1}(b) = 2/b$ for $b \in [0, 11/15]$, $Q_{B_1}(\tau) = 11\tau/15$ for $\tau \in [0, 1]$, and $\rho_V(v) = 1/v$ for $v \in [0, 1]$. Moreover, $\overline{H}(x; F, C_\perp) = x - \int_0^x F(t)dt/[3F(x)]$ and $\overline{L}(x; F, C_\perp) = x - 2\int_0^x F(t)dt/[3F(x)]$. The Volterra integral equation becomes

$$Q(\tau) + \frac{8}{7\tau}\int_0^\tau Q(s)ds = \frac{11\tau}{7},$$

with the unique solution $Q_o(\tau) = \tau, \tau \in [0, 1]$.

4 Estimation

In this section, we consider estimation of the model primitives, that is, the copula function and the quantile function of private signals. We focus on pure common value auction models without private-value bidders. To estimate the quantile function of private signals, we first need to estimate the kernel functions $k_{jo}(\tau, s)$ and the functions $\phi_{jo}(\tau)$ for $j = 1, 2$. Recall that $z_{1o}(\tau, s)$ and $z_{2o}(\tau, s)$ appear in the kernel functions and different partial derivatives of the copula function are involved in the definitions of $z_{1o}(\tau, s)$ and $z_{2o}(\tau, s)$. In principle, we could estimate the copula function and its partial derivatives nonparametrically following an idea similar to the local polynomial estimation for a regression function. In practice, however, the number of common-value bidders can be greater than three and auction data typically observed is not very large for a given number of bidders.[7] This prevents us from

[7]For example, in the OCS wildcat auction data set which we will use in the empirical application, there are 217 auctions when $M = 2$ and 330 auctions when $M = 3$.

a fully nonparametric approach due to the curse of dimensionality. We thus consider a semiparametric approach. Specifically, we parameterize the copula function and leave the quantile function of private signals nonparametric.

4.1 Estimation of the Copula Function

First, we consider estimating the copula function and its partial derivatives. We make the following assumption.

Assumption (PC) The true copula function $C_o(\underline{u}) = C_o(u_1, \cdots, u_M)$ lies in a parametric family indexed by $\theta \in \Theta$, with the true parameter θ_o.

Under Assumption (PC), estimating the copula function reduces to estimating the parameter θ_o. Let $c(\underline{u}; \theta)$ be the copula density function with parameter θ and define

$$\mathcal{L}(\theta) = \sum_{\ell=1}^{L} \log c\left(\widehat{G}_{B_1}(B_{1\ell}), \cdots, \widehat{G}_{B_1}(B_{M\ell}); \theta\right),$$

where $\widehat{G}_{B_1}(b) = \frac{1}{1+L} \sum_{\ell=1}^{L} \mathbb{1}(B_{1\ell} \le b)$. Notice that $\widehat{G}_{B_1}(b)$ is $L/(1+L)$ times the usual empirical distribution function. This rescaling avoids difficulties caused by the potential unboundedness of $\log c(u_1, \cdots, u_M)$ when some of the u_ms approach one. Following Oakes (1994) and Genest, Ghoudi, and Rivest (1995), we can estimate θ_o by

$$\widehat{\theta}_L = \arg\max_{\theta \in \Theta} \mathcal{L}(\theta).$$

Genest, Ghoudi, and Rivest (1995) establish the root-n consistency and asymptotic normality of $\widehat{\theta}_L$ (see also Chen, Fan, and Tsyrennikov (2006) for a semiparametric efficient sieve estimator of θ_o). Let $C(\underline{u}; \theta) = C(u_1, \cdots, u_M; \theta)$, then we can estimate $C(\underline{u}; \theta_o)$ by $C(\underline{u}; \widehat{\theta}_L)$, estimate $z_{1o}(\tau, s) \equiv z_1(\tau, s; \theta_o)$ by $z_1(\tau, s; \widehat{\theta}_L)$, and estimate $z_{2o}(\tau, s) \equiv z_2(\tau, s; \theta_o)$ by $z_2(\tau, s; \widehat{\theta}_L)$, respectively.

One practical question is the choice of the copula family. In pure common value auction models, private signals are assumed to be positively correlated and the level of dependence can range from independent to perfectly positively correlated. In choosing the family of copula functions in practice, a good candidate is the Archimedean family, which is flexible in the sense that it can allow any level of positive dependence. We thus further make the following assumption.

Assumption (AC) The true copula function of the private signals is an Archimedean copula function with strictly decreasing, twice continuously differentiable generator function $\varphi_{\theta_o}(\cdot)$, with its inverse function $\varphi_{\theta_o}^{-1}(\cdot)$ completely monotone on $[0, \infty)$.[8]

Under Assumption (AC), the true copula function is of the form

$$C_o(\underline{u}) = \varphi_{\theta_o}^{-1}\Big[\sum_{m=1}^{M} \varphi_{\theta_o}(u_m)\Big], \tag{4.1}$$

where $\varphi_{\theta_o} : [0, 1] \to [0, \infty)$ with $\varphi_{\theta_o}(1) = 0$. Let $\varphi_{\theta i}(\underline{u}) = \varphi_\theta^{(i)}[\varphi_\theta^{-1}(\varphi_\theta(u_1)+\cdots+\varphi_\theta(u_M))], i = 1, 2, 3$, where $\varphi_\theta^{(i)}(t)$ denotes the i-th partial derivative of $\varphi_\theta(t)$ with respect to t. Straightforward calculation gives

$$C_1(\underline{u}) = \frac{\varphi_\theta^{(1)}(u_1)}{\varphi_{\theta 1}(\underline{u})}, C_{12}(\underline{u}) = -\frac{\varphi_\theta^{(1)}(u_1)\varphi_\theta^{(1)}(u_2)\varphi_{\theta 2}(\underline{u})}{[\varphi_{\theta 1}(\underline{u})]^3},$$
$$C_{123}(\underline{u}) = \frac{\varphi_\theta^{(1)}(u_1)\varphi_\theta^{(1)}(u_2)\varphi_\theta^{(1)}(u_3)}{[\varphi_{\theta 1}(\underline{u})]^4}\left[-\varphi_{\theta 3}(\underline{u}) + \frac{3[\varphi_{\theta 2}(\underline{u})]^2}{\varphi_{\theta 1}(\underline{u})}\right],$$

and $z_1(\tau, s)$, $z_2(\tau, s)$ can be calculated accordingly.

In addition to the copula parameter θ under Assumption (PC), certain dependence measures are of their own interest as well in practice. For example, in the U.S. OCS wildcat auction, one might be interested in the level of dependence among the private signals which could reflect the preciseness of technology in conducting the seismic survey. In automobile auctions, the level of dependence among the private signals could reflect how widely the market information is diffused among the economic agents. Common measures of dependence level such as Kendall's τ_k and Spearman's ρ are closely related to the copula parameter θ. Consider Kendall's τ_k, where one version of the multivariate Kendall's τ_k is simply the average of pairwise Kendall's τ_k.[9] That is,

$$\tau_k(X_1, \cdots, X_M) = \frac{1}{\binom{n}{2}} \sum_{1 \le i < j \le M} \tau_k(X_i, X_j).$$

In the symmetric bidders' case, the multivariate Kendall's τ_k is simply $\tau_k(X_i, X_j)$ for any pair (X_i, X_j). Under Assumption (AC), it is known from Corollary 5.1.4 in Nelsen (2006) that

$$\tau_k(\theta) = 1 + 4\int_0^1 \frac{\varphi_\theta(t)}{\varphi'_\theta(t)} dt.$$

[8]Complete monotonicity of $\varphi_{\theta_o}^{-1}(\cdot)$ is a sufficient condition to guarantee that the expression in (4.1) actually generates an M-dimensional Archimedean copula function for any $M \ge 3$. This assumption suffices for the purpose of this paper. For a necessary and sufficient condition on this issue, see McNeil and Neslehova (2009).

[9]For an alternative Kendall's τ_k formula in the multivariate case, see Genest, Nešlehová, and Ghorbal (2011).

The copula generator functions and the relation between Kendall's τ_k and θ for Clayton, Frank, and Gumbel families are summarized in Table 1 and Figure 2. Given estimate $\widehat{\theta}_L$ of θ_o, we can estimate the level of dependence among the private signals.

	$\varphi_\theta(t)$	Kendall's τ_k
Clayton	$\frac{t^{-\theta}-1}{\theta}$	$\frac{\theta}{\theta+2}$
	$\theta \in [0, \infty)$	$-$
Frank	$-\log \frac{e^{-\theta t}-1}{e^{-\theta}-1}$	$\frac{\theta-4}{\theta} + \frac{4}{\theta^2}\int_0^\theta \frac{t}{e^t-1}dt$
	$\theta \in [0, \infty)$	$-$
Gumbel	$(-\log t)^\theta$	$\frac{\theta-1}{\theta}$
	$\theta \in [1, \infty)$	$-$

Table 1: Generator and Kendall's τ_k

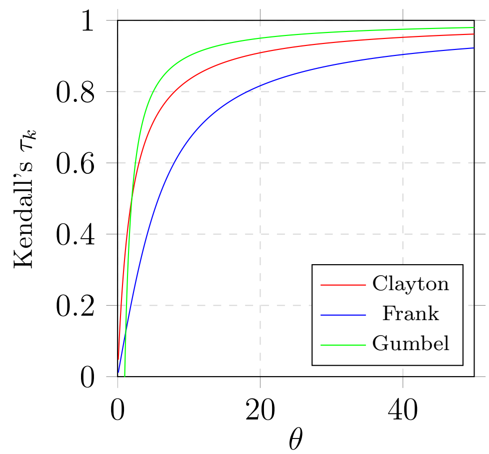


Figure 2: Kendall's τ_k and Parameter θ

The ranges of $\tau_k(\theta)$ are $[0, 1]$. Hence the Archimedean copula families are rich enough to accommodate any level of positive dependence for empirical works. In sum, Assumptions (PC) and (AC) reduce the dimension of the space of copula functions without much loss of generality. In a different context, Fan and Liu (2013) demonstrated that the estimation results are robust against mis-specification of the copula family (see also Zheng and Klein (1995), Huang and Zhang (2008), Chen (2010), and Hubbard, Li, and Paarsch (2012)). We will also show this robustness in both the simulation and the empirical application.

4.2 Estimation of the Signal Quantile Function

In this section, we consider estimation of the true quantile function $Q_o(\tau)$ of the private signals. The first-price and second-price pure common value auction models in Sections 3.1 and 3.2 induce the same linear inverse problem. That is, $Q_o(\tau)$ is subject to the restriction

$$(I - K_{jo})Q_o(\tau) = \phi_{jo}(\tau), j = 1, 2,$$

where I is the identity operator and $K_{jo}Q(\tau) = \int_0^\tau k_{jo}(\tau, s)Q(s)ds$, $j = 1, 2$, is a linear operator. For an excellent review on the linear inverse problem in structural econometrics, see Florens (2003) and Carrasco, Florens, and Renault (2007). Under Assumptions (CU-1) or (CU-2), the Volterra integral operator K_{jo} does not have nonzero spectral values. This implies that 1 is not an eigenvalue (see Kress (1999)). Therefore, $I - K_{jo}$ is invertible and it admits a linear continuous inverse $(I - K_{jo})^{-1}$. Unlike the linear inverse problem encountered in nonparametric instrumental regression problem such as that in Darolles, Fan, Florens, and Renault (2011), our linear inverse problem is well-posed and regularization is not needed.

Denote the estimator of K_{jo} as $\widehat{K}_j$, where

$$\widehat{K}_j Q(\tau) = \int_0^\tau \widehat{k}_j(\tau, s) Q(s) ds,$$

and $\widehat{k}_j(\tau, s)$ is the plug-in estimator of $k_{jo}(\tau, s)$ given estimator $\widehat{\theta}_L$ of θ_o. Let the estimator of $\phi_{jo}(\tau)$ be $\widehat{\phi}_j(\tau)$, then the estimator of $Q_o(\tau)$ is defined as

$$\widehat{Q}(\tau) = (I - \widehat{K}_j)^{-1} \widehat{\phi}_j(\tau). \tag{4.2}$$

Intuitively, if $\widehat{K}_j$ is close to K_{jo} w.p.a.1, then $(I - \widehat{K}_j)^{-1}$ is close to $(I - K_j)^{-1}$ w.p.a.1. The eigenvalues of $\widehat{K}_j$ should be close to the eigenvalues of K_{jo}. Therefore, 1 is not an eigenvalue of $\widehat{K}_j$ w.p.a.1. and $(I - \widehat{K}_j)^{-1}$ is continuous w.p.a.1. If $\widehat{\phi}_j$ is also close to ϕ_{jo}, then $(I - \widehat{K}_j)^{-1} \widehat{\phi}_j(\tau)$ should be close to $Q_o(\tau)$.

Consider first the estimation of $Q_o(\tau)$ in the first-price sealed-bid pure common value auction model. For this model, recall that

$$K_{1o} Q(\tau) = -\frac{(M-2)}{2} \int_0^\tau z_{1o}(\tau, s) Q(s) ds, \ \phi_{1o}(\tau) = \frac{M}{2} \left(Q_{B_1}(\tau) + \frac{1}{\rho_{M_1|B_1}(Q_{B_1}(\tau))} \right).$$

We use the standard empirical quantile function to estimate $Q_{B_1}(\tau)$, $\widehat{Q}_{B_1}(\tau) = B_{1\lceil L\tau \rceil}$, where $\lceil a \rceil$ is the smallest integer greater than or equal to a. For the estimation of $\rho_{M_1|B_1}(b)$, we follow Li, Perrigne, and Vuong (2002) and Haile, Hong, and Shum (2006) to use

$$\widehat{\rho}_{M_1|B_1}(b) = \frac{\widehat{G}_{M_1 \times B_1}(b)}{\widehat{g}_{M_1 B_1}(b)},$$

where

$$\widehat{G}_{M_1 \times B_1}(b) = \frac{1}{L} \sum_{\ell=1}^{L} \frac{1}{M} \sum_{m=1}^{M} \mathbb{1}(M_{m\ell} \le b) k_{G,h_G}(B_{m\ell} - b),$$
$$\widehat{g}_{M_1 B_1}(b) = \frac{1}{L} \sum_{\ell=1}^{L} \frac{1}{M} \sum_{m=1}^{M} k_{g,h_g}(M_{m\ell} - b) k_{g,h_g}(B_{m\ell} - b),$$

and $k_{G,h_G}(x) = k_G(\frac{x}{h_G})/h_G$, $k_{g,h_g}(x) = k_g(\frac{x}{h_g})/h_g$. Here, $k_G(\cdot)$ and $k_g(\cdot)$ are two kernel density functions and h_G, h_g are two bandwidth sequences. Let

$$\widehat{K}_1 Q(\tau) = -\frac{M-2}{2} \int_0^\tau z_1(\tau, s; \widehat{\theta}_L) Q(s) ds, \ \widehat{\phi}_1(\tau) = \frac{M}{2} \Big(\widehat{Q}_{B_1}(\tau) + \frac{1}{\widehat{\rho}_{M_1|B_1}(\widehat{Q}_{B_1}(\tau))} \Big).$$

We estimate $Q_o(\tau)$ by

$$\widehat{Q}(\tau) = (I - \widehat{K}_1)^{-1} \widehat{\phi}_1(\tau)$$

in the first-price sealed-bid pure common value auction model. We make the following regularity assumptions.

Assumption (KS) $k_G(\cdot), k_g(\cdot)$ are symmetric density functions with bounded support and have continuous bounded first derivatives.

Assumption (BS) $h_G = c_G(\log L/L)^{\frac{1}{2d+2M-3}}$, $h_g = c_g(\log L/L)^{\frac{1}{2d+2M-2}}$ for some constants c_G, c_g, and d is the differentiability order of the private signals' joint distribution.

Assumption (UB)
(i) $\rho_{M_1|B_1}(b)$ is uniformly bounded away from zero on $[\underline{b}, \overline{b}]$;
(ii) $g_{M_1B_1}(b)$ is uniformly bounded away from zero on $[\underline{b}, \overline{b}]$;
(iii) $G_{M_1\times B_1}(b) = \partial G_{M_1B_1}(m_1, b_1)/\partial b_1|_{m_1=b_1=b}$ is uniformly bounded from above on $[\underline{b}, \overline{b}]$.

Assumption (RA-1) Given $\epsilon > 0$,
(i) $w_{1\epsilon}(\theta) = \sup\limits_{t\in[\epsilon,1-\epsilon]} \int_0^1 |z_1(t,s;\theta) - z_{1o}(t,s)|ds$ is continuous at θ_o;
(ii) $w_{2\epsilon}(\theta) = \sup\limits_{t\in[\epsilon,1-\epsilon]} \int_0^1 |z_1(t,s;\theta) - z_{1o}(t,s)|Q_o(s)ds$ is continuous at θ_o;
(iii) $||(I-K_{1o})^{-1}||_\epsilon = \sup_{||\varphi||_\epsilon \neq 0} ||(I-K_{1o})^{-1}\varphi||_\epsilon/||\varphi||_\epsilon < \infty$, where $||\varphi||_\epsilon = \sup_{t\in[\epsilon,1-\epsilon]} |\varphi(t)|$.

Assumption (RA-2) $w_{10}(\theta), w_{20}(\theta)$ are continuous at θ_o for $w_{1\epsilon}(\theta), w_{2\epsilon}(\theta)$ in Assumption (RA-1) evaluated at $\epsilon = 0$.

Theorem 4.1 In a first-price sealed-bid pure common value auction, under Assumptions (PC), (KS), (BS), (UB), and (RA-1),

$$\sup_{\tau\in[\epsilon,1-\epsilon]} \left|\widehat{Q}(\tau) - Q_o(\tau)\right| = o_P(1).$$

Proof. See the Appendix. ■

Next, we consider estimating $Q_o(\tau)$ in the second-price sealed-bid pure common value auction model. In this model, recall that

$$K_{2o}Q(\tau) = -\frac{M-2}{2}\int_0^\tau z_{1o}(\tau,s)Q(s)ds,\ \phi_{2o}(\tau) = \frac{MQ_{B_1}(\tau)}{2}.$$

We estimate $Q_o(\tau)$ by

$$\widehat{Q}(\tau) = (I - \widehat{K}_2)^{-1}\widehat{\phi}_2(\tau),$$

where

$$\widehat{K}_2Q(\tau) = -\frac{M-2}{2}\int_0^\tau \widehat{z}_1(\tau,s;\widehat{\theta}_L)Q(s)ds,\ \widehat{\phi}_2(\tau) = \frac{M\widehat{Q}_{B_1}(\tau)}{2}.$$

Theorem 4.2 In a second-price sealed-bid pure common value auction, under Assumptions (CU-2), (PC), and (RA-2),

$$\sup_{\tau\in[0,1]}\left|\widehat{Q}(\tau)-Q_o(\tau)\right|=o_P(1).$$

Proof. See the Appendix. ■

5 Simulation

In this section, we conduct a Monte Carlo simulation to evaluate the finite sample performance of our estimator. We focus on the first-price sealed-bid pure common value auction in both the simulation and the empirical application. In the simulation designs, we set the number of bidders $M=3$ and let the true copula function be the independent copula. For the marginal distribution function of private signals, we follow Marmer and Shneyerov (2012) to let $F_o(x)=x^\alpha, x\in[0,1]$ for $\alpha=0.5,1,2$. Table 5 summarizes the true quantile functions and equilibrium bidding functions in each design. Three different sample sizes, $L=100,200$, and 500, are used. The number of repetitions is set to be 1000.

α	$Q_o(\tau)$	$\beta(x)$
0.5	τ^2	$\frac{7}{72}(8x-4\sqrt{x}-e^{-4\sqrt{x}}+1)$
1	τ	$5x/9$
2	$\tau^{\frac{1}{2}}$	$\frac{16}{9}e^{\frac{2}{x}}\int_{2/x}^{\infty}\frac{e^{-t}}{t}dt$

Table 2: Equilibrium Bidding Functions under Different Marginal Distributions

In estimating the copula parameter, we employ three popular copula families, namely the Clayton, Frank, and Gumbel families. Each of them nests the independent copula as a special case. In estimating the observed conditional reserve hazard function, we follow Li, Perrigne, and Vuong (2002) and use the triweight kernel for both $k_G(\cdot)$ and $k_g(\cdot)$,

$$k_G(s)=k_g(s)=\frac{35}{32}(1-s^2)^3\mathbb{1}(|s|\leq 1).$$

Bandwidth choice of h_G,h_g follows from Assumption (BS) with $d=3$. The constants in bandwidth are set to be $c_G=c_g=2.978\times 1.06\widehat{\sigma}_B$ due to our choice of triweight kernel, where $\widehat{\sigma}_B$ is the empirical standard deviation of the bids.

For estimation of the quantile function, using the relation $Q_o(\tau)=(I-K_{1o})^{-1}\phi_{1o}(\tau)$ and replacing the unknown true quantities with their estimators, we get

$$\widehat{Q}(\tau)=(I-\widehat{K}_1)^{-1}\widehat{\phi}_1(\tau)=\sum_{j=0}^{\infty}\widehat{K}_1^j\widehat{\phi}_1(\tau). \tag{5.1}$$

We name this geometric series estimator (GSE).[10] In the implementation, let $\widehat{Q}^{(J)}(\tau) = \sum_{j=0}^{J} \widehat{K}_1^j \widehat{\phi}_1(\tau)$, and we use the following convergence criterion: stop the iteration if

$$\frac{\sum_{p=1}^{P} \left[\widehat{Q}^{(J+1)}(x_p) - \widehat{Q}^{(J)}(x_p)\right]^2}{\sum_{p=1}^{P} \left[\widehat{Q}^{(J)}(x_p)\right]^2 + 0.0001} < 0.001,$$

where $x_p, p = 1, \cdots, P$ are P evaluation points. Previous works using this criterion include Nielsen and Sperlich (2005), Henderson et al. (2008), Mammen et al. (2009), and Su and Lu (2013). Following Su and Lu (2013), we choose 100 evaluation points evenly distributed between the 0.2 and 0.8 quantiles of the private signals. The evaluation points are fixed across repetitions. In our simulation designs, the geometric series estimator typically hits the convergence criterion when $J = 3$.

Furthermore, under Assumption (CU-1), $Q_o(\tau)$ is the unique minimizer of

$$\mathcal{M}(Q) = \int_0^1 [(I - K_{1o})Q(\tau) - \phi_{1o}(\tau)]^2 d\tau.$$

This suggests another estimator of $Q_o(\tau)$, namely, $\widetilde{Q}(\tau) = \arg\min_Q \widehat{\mathcal{M}}(Q)$, where

$$\widehat{\mathcal{M}}(Q) = \int_0^1 \left[(I - \widehat{K}_1)Q(\tau) - \widehat{\phi}_1(\tau)\right]^2 d\tau. \tag{5.2}$$

In principle, a weighting function in the definition of $\widehat{\mathcal{M}}(Q)$ can be used for different τ to improve efficiency, we choose the unweighted criterion as a baseline and leave the proper choice of weighting function to future research. We estimate $Q_o(\tau)$ by sieve method (see Chen (2007) for an excellent review on the sieve method).

Note that the quantile function is defined on $[0, 1]$, we can use the Bernstein polynomial sieve basis in the simulation. A Bernstein polynomial sieve of order H_L is defined as

$$B_{H_L}(t) = \sum_{j=0}^{H_L} \alpha_j \binom{H_L}{j} t^j (1-t)^{H_L - j}, t \in [0, 1],$$

with Bernstein cofficients $\alpha_j, j = 1, \cdots, H_L$. In the implementation, we follow Gentry, Li, and Lu (2015) to approximate the integral in (5.2) by specifying a discrete grid $g_\tau \subset [0, 1]$ and use the discretized criterion function

$$\widehat{\mathcal{M}}^d(Q) = \sum_{\tau \in g_\tau} \left[(I - \widehat{K}_1)Q(\tau) - \widehat{\phi}_1(\tau)\right]^2.$$

[10] In the implementation of both the geometric series estimator and the iterative sieve estimator below, computing multiple integral is needed. We approximate the integral by dividing the interval $[0, 1]$ into 100 subintervals and compute the discretized sum.

Due to the bias of $\widehat{\phi}_1(\tau)$ on the boundary, we use a grid with 100 evenly spaced points between the 0.05 and 0.95 quantiles of the private signals. In both the simulation and the empirical application, we experimented with different orders and found that the estimator when $H_L = 2$ performs best.

Let $\widetilde{Q}^{(0)} = \arg\min_Q \widehat{\mathcal{M}}(Q)$. Note that $Q_o(\tau) = K_{1o}Q_o(\tau) + \phi_{1o}(\tau)$, under conditions such that the geometric series estimator is convergent, the sequence of approximations

$$Q^{(J)}(\tau) = K_{1o}Q^{(J-1)}(\tau) + \phi_{1o}(\tau), J = 1, 2, \cdots,$$

is close to $Q_o(\tau)$ from any starting point $Q^{(0)}(\tau)$. In addition, if $\widehat{K}_1$ and $\widehat{\phi}_1(\tau)$ are sufficiently close to K_{1o} and $\phi_{1o}(\tau)$, respectively, then

$$\widetilde{Q}^{(J)}(\tau) = \widehat{K}_1\widetilde{Q}^{(J-1)}(\tau) + \widehat{\phi}_1(\tau), J = 1, 2, \cdots \tag{5.3}$$

is close to $\widehat{Q}(\tau)$. We name this iterative sieve estimator (ISE). The same convergence criterion as in the geometric series estimator is used and the iterative sieve estimator typically hits the convergence criterion when $J = 2$ in our simulation designs.

L	100		200		500	
Clayton	GSE	ISE	GSE	ISE	GSE	ISE
25% quantile	0.0295	0.0278	0.0235	0.0209	0.0179	0.0146
50% quantile	0.0613	0.0597	0.0487	0.0463	0.0379	0.0349
75% quantile	0.1344	0.1330	0.1230	0.1211	0.1147	0.1125
Frank						
25% quantile	0.0295	0.0277	0.0235	0.0209	0.0178	0.0145
50% quantile	0.0617	0.0601	0.0490	0.0466	0.0381	0.0351
75% quantile	0.1336	0.1322	0.1224	0.1205	0.1143	0.1121
Gumbel						
25% quantile	0.0295	0.0278	0.0234	0.0209	0.0178	0.0145
50% quantile	0.0618	0.0602	0.0490	0.0467	0.0381	0.0351
75% quantile	0.1343	0.1330	0.1234	0.1215	0.1151	0.1128

Table 3: RMSE of GSE and ISE, $\alpha = 0.5$

In Tables 3, 4, 5, we report the estimated RMSEs of both the geometric series estimator and the iterative sieve estimator at different quantiles of the private signals. We focus on discussing the simulation results in Table 4. In this case, $\alpha = 1$ and the true quantile function is the identity function $Q_o(\tau) = \tau, \tau \in [0, 1]$. First, for the same sample size, the

L	100		200		500	
Clayton	GSE	ISE	GSE	ISE	GSE	ISE
25% quantile	0.0457	0.0453	0.0319	0.0320	0.0219	0.0223
50% quantile	0.0479	0.0486	0.0329	0.0334	0.0227	0.0233
75% quantile	0.0917	0.0931	0.0778	0.0788	0.0596	0.0607
Frank						
25% quantile	0.0456	0.0453	0.0319	0.0320	0.0219	0.0223
50% quantile	0.0477	0.0484	0.0328	0.0333	0.0227	0.0232
75% quantile	0.0909	0.0924	0.0773	0.0784	0.0592	0.0603
Gumbel						
25% quantile	0.0456	0.0453	0.0318	0.0319	0.0219	0.0223
50% quantile	0.0477	0.0483	0.0328	0.0333	0.0227	0.0232
75% quantile	0.0916	0.0931	0.0782	0.0793	0.0601	0.0613

Table 4: RMSE of GSE and ISE, $\alpha = 1$

L	100		200		500	
Clayton	GSE	ISE	GSE	ISE	GSE	ISE
25% quantile	0.0610	0.0618	0.0486	0.0511	0.0431	0.0452
50% quantile	0.0754	0.0703	0.0690	0.0644	0.0696	0.0641
75% quantile	0.0445	0.0467	0.0318	0.0336	0.0230	0.0194
Frank						
25% quantile	0.0617	0.0617	0.0492	0.0512	0.0436	0.0453
50% quantile	0.0749	0.0691	0.0687	0.0636	0.0695	0.0636
75% quantile	0.0442	0.0462	0.0318	0.0334	0.0233	0.0194
Gumbel						
25% quantile	0.0617	0.0617	0.0492	0.0512	0.0436	0.0454
50% quantile	0.0748	0.0691	0.0686	0.0636	0.0694	0.0636
75% quantile	0.0438	0.0462	0.0314	0.0334	0.0225	0.0190

Table 5: RMSE of GSE and ISE, $\alpha = 2$

estimated RMSEs are very close between the geometric series estimator and the iterative sieve estimator. Consider the median RMSE for $L = 100$ for example, the two estimated RMSEs are 0.0479 and 0.0486 for the Clayton copula, 0.0477 and 0.0484 for the Frank copula, and 0.0477 and 0.0483 for the Gumbel copula. Second, the robustness against copula specification is confirmed in our simulation. The estimated RMSEs are very close across different specifications of the copula families. Consider the median RMSE when $L = 200$. For the geometric series estimator, the estimated RMSEs are $0.0329, 0.0328, 0.0328$ for the Clayton, Frank and Gumbel copula families, respectively. For the iterative sieve estimator, the estimated RMSEs are $0.0334, 0.0333, 0.0333$ for the three copula families, respectively. Third, as the sample size increases, the estimation precision increases. For example, when the Clayton copula is used, the estimated median RMSEs of the geometric series estimator are $0.0479, 0.0329$, and 0.0227 when $L = 100, 200$, and 500, while the estimated median RMSEs of the iterative sieve estimator are $0.0486, 0.0334$, and 0.0233 when $L = 100, 200$, and 500. The cases when $\alpha = 0.5$ and 2 follow similar patterns and discussions.

6 Empirical Application

6.1 Data Description

In this section, we study the U.S. OCS wildcat auction in the pure common value auction framework. The United States federal government has been selling gas and oil exploration rights on offshore lands off the coasts of Texas and Louisiana to the private sector since 1954. There are three types of oil and gas lease sales: wildcat sales, developmental sales, and drainage sales. A wildcat sale refers to tracts located in previously unexplored areas with unknown geological and seismic characteristics. A developmental sale refers to tracts previously sold but re-offered due to either the government's rejection of the winning bid or the winner's relinquishment. A drainage sale is the sale of tracts in which oil or gas deposits have been found. Among the three types, the wildcat auction fits into the symmetric pure common value framework (Hendricks, Pinkse, and Porter (2003)) for two reasons. First, the future selling price of gas or oil from one tract is the same for different bidders. Second, the exact volume of deposit is unknown to each bidder and no bidder has more information than others, thus bidders can be approximately viewed as symmetric. Therefore, we focus on the wildcat auction.

One tract is sold in each wildcat auction. A tract is defined as either a block or half a block and a block is usually either 5000 or 5760 acres of land. Potential participants in wildcat sales are allowed to carry out a seismic investigation before the sale date, but they are not allowed to drill any exploratory wells. Each firm evaluates the tracts by analyzing its seismic survey. This provides noisy but roughly equally informative signals about the amount of oil and gas on a lease. In a given sale, all of the announced tracts are sold simultaneously by first-price sealed-bid auctions. The Department of the Interior announces the values of all submitted bids and identities of the firms. The owner of the lease has to pay a nominal rental fee until the production begins, normally \$3 per acre per year for wildcat tracts (Porter, 1995). A fixed portion of the revenue is claimed by the government as royalty payment.

In the wildcat auction, there may be a reservation price of \$15 or \$25 per acre (Porter, 1995). The reservation price is the same across all tracts within a sale but may be different across sales. The reserve price has long been perceived as too low (see McAfee and Vincent (1992)), thus we follow Li, Perrigne, and Vuong (2000) to view the bids as from auctions with non-binding reserve price. Given our nonparametric identification of the seller's expected profit under any reserve price, we are interested in whether or not the actual reserve price is indeed too low, and if it is, what is the optimal reserve price that could have generated more revenue for the government.

$M = 3$	
#Tracts	254
Mean	4.932
Median	2.412
Min Bid	0.209
Max Bid	38.244
Std	6.338
Million in 1982\$ per Tract	

Table 6: Summary Statistics for Bids

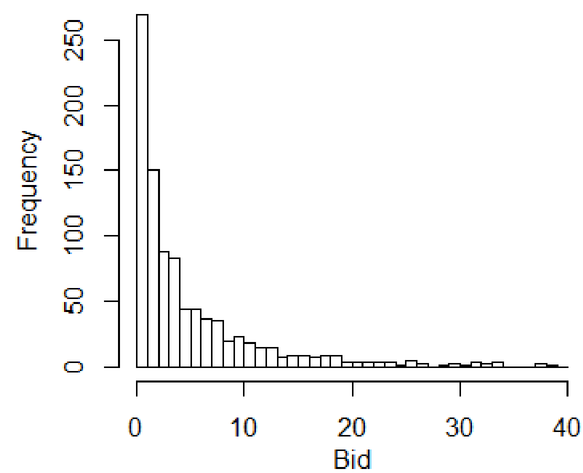


Figure 3: Histogram of Bids

The data is obtained from the website of the Center for the Study of Auctions, Procurements and Competition Policy at Penn State. We focus on a subset of the auctions with three bidders. The original data set includes 330 auctions with three bidders, where the maximum bid is 211.89 million dollars per tract in 1982\$ and minimum bid is merely 0.000543 million dollars. To avoid the possible contamination of these outliers, we trim the data so that auctions with bids smaller than a certain lower threshold level or larger than a certain upper threshold level are dropped.[11] This leaves us with 254 auctions. Table 6 and Figure 3 provide some descriptive statistics of the data set. In particular, the bids are concentrated to

[11]This trimming makes the auctions in the data set more homogenized. However, this results in a trade-off between more homogenized auctions and smaller sample sizes. After some preliminary analysis, we set the lower threshold to be 0.2 million dollars, and the upper threshold level to be 40 million dollars.

the left in Figure 3 and the histogram implies that the density function of bids decreases as we move towards the right. Given that bids are strictly increasing transformation of private signals, we expect the density of private signal to have similar pattern.

6.2 Empirical Findings

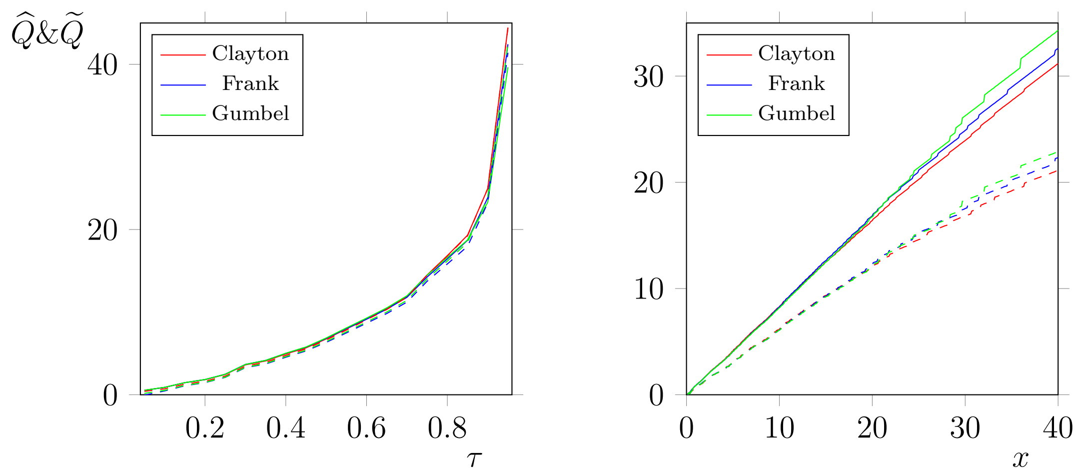
In the OCS wildcat auction, each bidder has partial information on the unknown exact volume of deposit and their partial information are correlated. First, when firms jointly hire a geophysical company to shoot the seismic survey of a tract, although different firms may have different algorithms to analyze and interpret the survey data, it is expected that their algorithm outputs are correlated. Second, even when each firm conducts its own seismic survey, given similar technologies, the estimates of oil volume of the same area or tract from different firms should be correlated since each one is an estimate of the true volume. It is therefore of interest to know whether their private signals are correlated and to what degree they are correlated. To this end, we estimate the copula parameter θ as well as Kendall's τ_k and summarize results in Table 7. The estimated Kendall's τ_ks are around 0.2 for different copula families and the 95% confidence intervals reveal that the Kendall's τ_k is statistically significantly different from zero, suggesting a positive dependence among the private signals.

	Clayton	Frank	Gumbel
θ	0.469	2.027	1.259
	$(0.320, 0.619)$	$(1.759, 2.295)$	$(1.184, 1.334)$
τ_k	0.190	0.217	0.206
	$(0.140, 0.241)$	$(0.190, 0.243)$	$(0.159, 0.253)$

Table 7: Estimated θ and Kendall's τ_k with 95% Bootstrap Confidence Interval

Next, we estimate the quantile function of private signals. As in the simulation, we employ the two methods (GSE and ISE) and the three Archimedean copula families (Clayton, Frank, and Gumbel). We are interested in whether the estimation results are robust against the choice of copula family. This is important since fully nonparametric estimation of high dimensional copula and its partial derivatives would be difficult in practice due to the curse of dimensionality.

The estimated curves are shown in Panel (a) of Figure 4. The geometric series estimates are solid lines and the iterative sieve estimates are dashed lines. The estimated curves almost coincide with each other. The robustness against estimation method is expected since in theory, the two estimators should converge to the same function in the limit. The robustness against choice of copula family suggests that little loss is incurred when we parameterize the copula function in practice. From the estimated quantile functions, the upper support of private signals is around 40 million dollars. The first quartile, median and third quartile are around 2.3, 6.5 and 14.5 million dollars with interquartile range to be around 12.2 million dollars. This implies that the private signals have a large probability of taking small values. The slope of the estimated quantile functions are strictly increasing almost everywhere when τ increases, implying that the density function of signal decreases as we move towards the right. This is in line with the histogram of bids in Figure 3 since bids are strictly increasing transformation of private signals.



(a) Quantile Function

(b) Conditional Expected Valuation

Figure 4: Estimates of Signal Quantile and Conditional Expected Valuation Functions

Given the estimated copula function and the quantile function, we are ready to estimate the two conditional expected valuation functions. The estimated curves are shown in Panel (b) of Figure 4. Solid lines represent the estimates of high conditional expected valuation and the dashed lines represent the estimates of low conditional expected valuation.[12] The estimated curves are very close to each other when different copula families are used, especially when signal is below 20 (or roughly 90% percentile of the estimated distribution).

Based on the closeness of the two estimated conditional expected valuation functions, we expect the robustness of estimated policy parameters against the choice of the copula family

[12]Given the robustness of estimated quantile function against estimation method, we show the estimated conditional expected valuation functions with the quantile function estimated from the iterative sieve method in Panel (b) of Figure 4, the estimates of conditional expected valuation functions with the quantile function estimated from the geometric series method are very similar and thus omitted to save space.

and present the results as follows.

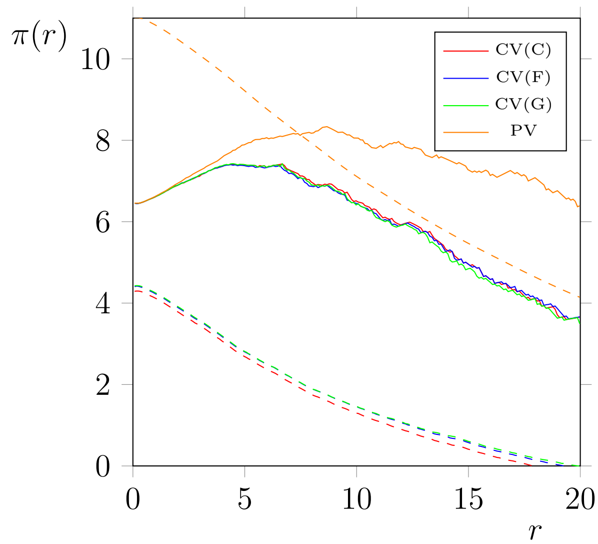


Figure 5: Estimated Seller's Expected Profit and Bidders' Expected Surplus (ISE)

	Optimal Reserve		Max Profit	
	GSE	ISE	GSE	ISE
Clayton	4.27	4.41	7.38	7.42
	–	–	(14.2%)	(14.8%)
Frank	4.39	4.42	7.38	7.41
	–	–	(14.2%)	(14.7%)
Gumbel	4.45	4.46	7.41	7.43
	–	–	(14.7%)	(15.0%)
PV	8.69		8.33	
	–		(28.9%)	
Actual	0.075/0.125		6.46	

Table 8: Optimal Reserve Price and Maximized Profit (Million $ per Tract)

First, the estimated curves of policy parameters are shown in Figure 5 using the iterative sieve estimator (estimated curves from the geometric series estimator are very similar thus omitted). In Figure 5, "C" represents Clayton copula, "F" represents Frank copula, and "G" represents Gumbel copula, the solid lines represent the seller's expected profit and the dashed lines represent the bidders' expected surplus.[13] The estimated curves of the seller's expected profit peak at a point above v_o, which is consistent with the theoretical prediction that a reserve price higher than v_o can be used as a screening tool to attract bidders with higher private signals and thus generate a higher expected profit. When reserve price is greater than the optimal reserve price, the seller's expected profit decreases since too few bidders will actually participate in the auction. The estimated curve of the bidders' expected surplus decreases from the beginning. This is also consistent with the theory that it should be maximized when there is no reserve price. The estimated curves are very close to each other.

Second, we present the results of optimal reserve price and maximized profit in Table 8. The estimated optimal reserve prices are between 4.27 and 4.46 million dollars per tract, which are quite close when different estimators and/or different copula families are used. The actual reserve price is 0.075/0.125 million dollars per tract (or equivalently 15/25$ per acre), which has been long perceived to be too low. Using our estimates, if the optimal reserve price were used, the government's revenue would be between 7.38 and 7.43 million

[13]Notice that $r \leq 20$ in Figure 5, this is because the maximum reserve price equals $\overline{L}(\overline{x})$ (otherwise all bidders will be screened out), which is around 20 from Panel (b) of Figure 4.

dollars per tract, which amounts to an increase between 14.2% and 15.0% upon the actual revenue.[14]

Lastly, for comparison purposes, we also estimate the two policy parameters under the private value framework. The estimated curves are also shown in Figure 5, which are quite different from those estimates in the pure common value framework. In the private value framework, the optimal reserve price is 8.68, with the maximized profit 8.33, which amounts to a 28.9% increase over the actual profit. The estimated maximum profits in the pure common value framework are more conservative than that in the private value framework.[15] If the private value framework is used to guide the choice of optimal reserve price, the profit will only increase by 6.8% upon the actual profit, leading to an loss of 134 million dollars compared to the maximized revenue that our optimal reserve price can generate. This comparison re-emphasizes the important implications of model specification on the policy parameters. In practice, if it is uncertain which framework is more appropriate, policy makers are suggested to estimate these policy parameters under both the common value framework and the private value framework and use the results as complements.

7 Concluding Remarks

In this paper, we study the identification problem in the pure common value auction models. We analyze the main challenges in the nonparametric identification of the full joint distribution of the common value and the private signals. We argue that identifying the full joint distribution is sufficient but not necessary for certain policy parameters. In particular, we show that for the identification of the seller's expected profit and the bidders' expected surplus under any reserve price, information on the two conditional expected valuation

[14]Alternatively, it is easy to see that the expected total welfare, defined as the sum of seller's expected profit and bidders' expected surplus, is maximized when the reserve price is set to be the seller's own valuation v_o. Consequently, any reserve price above v_o would incur a loss in the expected total welfare. If the reserve price is set to maximize the seller's expected profit, then the bidders would pay a higher price on average. In this case, although the government's revenue increases, the higher price paid by the oil company would translate to a higher gasoline price that the consumers need to pay. Therefore, from a society's point of view, using the optimal reserve price might not be a good choice. In practice, the government could choose a reserve price above v_o but below the optimal reserve price to balance the tradeoff between its own revenue and the expected total welfare. This tradeoff can be analyzed using the estimated seller's expected profit and bidders' expected surplus curves.

[15]Li, Perrigne, and Vuong (2003) focused on the two bidders' case in an affiliated private value framework, they find that the government's revenue could have increased by 50% if the optimal reserve price were employed.

functions are sufficient.

First, in both the first-price and second-price sealed-bid auction models, we establish nonparametric identification of the two conditional expected valuation functions under a weak assumption on the joint distribution of the common value and the private signals. Identifying them is essentially due to direct identification of the private signals' copula function from the observed bids' copula function and due to identification of the signal quantile function by a Volterra integral equation of a second kind. As a result, the seller's expected profit and the bidders' expected surplus under any reserve price are nonparametrically identified in both models. Second, we extend our approach to establish identification of the joint distribution of private signals in the second-price sealed-bid auction model with both common-value bidders and private-value bidders in Tan and Xing (2011). Third, we propose a semiparametric estimation method and establish consistency of the estimator for the signal quantile function. Monte Carlo experiment is conducted to show the satisfactory finite sample performances.

Lastly, we apply our estimation procedure to analyze the U.S. OCS wildcat auction data set. This data set has been perceived to better fit into the pure common value model than a private value one, but all structural estimation of this data set has been conducted in the private value framework. Our results suggest that the private signals are positively correlated, with the estimated Kendall's τ_k to be around 0.20. We estimate the seller's expected profit and the bidders' expected surplus to perform counterfactual analysis. We find that the actual reserve price is much lower than the optimal reserve price. Using our optimal reserve price can increase the U.S. government's revenue by 15.0%, which amounts to 246 million dollars for all the auctions considered in the sample. We also conduct similar analysis under the private value framework for comparison purposes. We find that the optimal reserve price in this framework is significantly different from that in the pure common value framework. If the private value framework is used to guide the choice of optimal reserve price, the government's revenue will only increase by 6.8% upon the actual profit, leading to an loss of 134 million dollars compared to the maximized revenue that our optimal reserve price can generate.

Several future research directions can be considered. First, although we assume that the common value is the simple average of the private signals up to some independent stochastic error, it is expected that the simple average can be generalized to other forms. In such situations, the quantile function of private signals is expected to be subject to a similar but possibly nonlinear Volterra integral equation. Given identification of the signal quantile function, our approach in identifying the seller's expected profit and the bidders' expected surplus applies. Second, although we focus on the expected value of the seller's profit and the bidders' surplus in this paper, it would be of interest to study the distribution function of these two quantities. In fact, from the derivation in the Appendix, the seller's profit under any reserve price, as a random variable, is a known function of the observed quantities. Similar discussion applies to the bidders' surplus in the case that the ex-post common value is observed. Third, the existence and uniqueness of Bayesian Nash equilibrium as well as econometric identification in a first-price sealed-bid auction with both common-value bidders and private-value bidders is of great interest. This problem is more challenging and the pair of equilibrium bidding functions is characterized by a system of differential equations. This is currently under investigation by the author.

APPENDIX

Proof. (Proposition 2.2) Let the seller's own valuation be v_o, denote the equilibrium bidding function under r as $\beta_r(x), x \in [x_r^*, \overline{x}]$ and the equilibrium bidding function without reserve price as $\beta(x), x \in [0, \overline{x}]$. The seller's profit under reserve price r is

$$\begin{aligned}
\pi_S(r) &= v_o \mathbb{1}\left(\beta_r(X^{(M)}) < r\right) + \beta_r\left(X^{(M)}\right) \mathbb{1}\left(\beta_r\left(X^{(M)}\right) \geq r\right) \\
&= v_o \mathbb{1}\left(X^{(M)} < x_r^*\right) + \beta_r\left(X^{(M)}\right) \mathbb{1}\left(X^{(M)} \geq x_r^*\right) \\
&\overset{(1)}{=} v_o \mathbb{1}\left(\beta(X^{(M)}) < \beta(x_r^*)\right) + \left(\beta(X^{(M)}) + (r - \beta(x_r^*)) J(x_r^*|X^{(M)})\right) \mathbb{1}\left(\beta(X^{(M)}) \geq \beta(x_r^*)\right) \\
&\overset{(2)}{=} v_o \mathbb{1}\left(B^{(M)} < b_r^*\right) + \left(B^{(M)} + (r - b_r^*) J^*(b_r^*|B^{(M)})\right) \mathbb{1}\left(B^{(M)} \geq b_r^*\right),
\end{aligned}$$

where $b_r^* = \beta(x_r^*)$, (1) follows from

$$\begin{aligned}
\beta_r(x) - \beta(x) &= \left[r - \overline{H}(x_r^*)\right] J(x_r^*|x) + \int_0^{x_r^*} J(a|x) d\overline{H}(a) \\
&= \left[r - \overline{H}(x_r^*)\right] J(x_r^*|x) + J(x_r^*|x) \int_0^{x_r^*} J(a|x_r^*) d\overline{H}(a) \\
&= \left[r - \overline{H}(x_r^*)\right] J(x_r^*|x) + J(x_r^*|x) \left[\overline{H}(x_r^*) - \beta(x_r^*)\right] \\
&= \left[r - \beta(x_r^*)\right] J(x_r^*|x), x \in [x_r^*, \overline{x}],
\end{aligned}$$

and (2) follows from

$$\begin{aligned}
J(x_r^*|x) &= \exp\left(-\int_{x_r^*}^{x} \rho_{Y_1|X_1}(s) ds\right) \\
&= \exp\left(-\int_{\beta(x_r^*)}^{\beta(x)} \rho_{M_1|B_1}(\beta(s)) \beta'(s) ds\right) \\
&= \exp\left(-\int_{\beta(x_r^*)}^{\beta(x)} \rho_{M_1|B_1}(t) dt\right) \equiv J^*(b_r^*|\beta(x)).
\end{aligned}$$

Taking expectation yields that the seller's expected profit under reserve price r is

$$\mathrm{E}[\pi_S(r)] = v_o \mathrm{E}[\mathbb{1}(B^{(M)} < b_r^*)] + \mathrm{E}[\pi_P(r)],$$

where $\mathrm{E}[\pi_P(r)] = \mathrm{E}\left[\left(B^{(M)} + (r - b_r^*) J^*(b_r^*|B^{(M)})\right) \mathbb{1}\left(B^{(M)} \geq b_r^*\right)\right]$.

Next, the bidders' surplus under reserve price r is

$$\begin{aligned}
\pi_B(r) &= \left(X_o - \beta_r(X^{(M)})\right) \mathbb{1}\left(\beta_r(X^{(M)}) \geq r\right) \\
&= X_o \mathbb{1}\left(\beta_r(X^{(M)}) \geq r\right) - \beta_r(X^{(M)}) \mathbb{1}\left(\beta_r(X^{(M)}) \geq r\right).
\end{aligned}$$

Then

$$
\begin{aligned}
&\mathrm{E}[\pi_B(r)]\\
&= \mathrm{E}\left[X_o \mathbb{1}\left(\beta_r(X^{(M)}) \geq r\right)\right] - \mathrm{E}[\pi_P(r)]\\
&= \mathrm{E}[X_o | X^{(M)} \geq x_r^*]\mathrm{E}[\mathbb{1}(X^{(M)} \geq x_r^*)] - \mathrm{E}[\pi_P(r)]\\
&= \left(\int_{x_r^*}^{\overline{x}} \mathrm{E}[X_o|X^{(M)} = x] dF_{X^{(M)}}(x|X^{(M)} \geq x_r^*)\right)\mathrm{E}[\mathbb{1}(X^{(M)} \geq x_r^*)] - \mathrm{E}[\pi_P(r)]\\
&= \int_{x_r^*}^{\overline{x}} \mathrm{E}[X_o|X^{(M)} = x] dF_{X^{(M)}}(x) - \mathrm{E}[\pi_P(r)]\\
&\overset{(1)}{=} \int_{x_r^*}^{\overline{x}} \int_0^x \overline{L}(x) f_{X_1Y_1}(u,x) du dx + \int_{x_r^*}^{\overline{x}} \int_0^x \overline{L}(x) f_{X_1Y_1}(x,u) du dx - \mathrm{E}[\pi_P(r)]\\
&\overset{(2)}{=} \int_{b_r^*}^{\bar{b}} \int_{\underline{b}}^{m_1} \overline{L}(\beta^{-1}(m_1)) g_{B_1M_1}(b_1,m_1) db_1 dm_1 + \int_{b_r^*}^{\bar{b}} \int_{\underline{b}}^{b_1} \overline{L}(\beta^{-1}(b_1)) g_{B_1M_1}(b_1,m_1) dm_1 db_1 - \mathrm{E}[\pi_P(r)]\\
&= \mathrm{E}\left[\mathbb{1}(M_1 \geq b_r^*)\mathbb{1}(B_1 \leq M_1)\overline{L}(\beta^{-1}(M_1))\right] + \mathrm{E}\left[\mathbb{1}(B_1 \geq b_r^*)\mathbb{1}(M_1 \leq B_1)\overline{L}(\beta^{-1}(B_1))\right] - \mathrm{E}[\pi_P(r)],
\end{aligned}
$$

where $\beta^{-1}(b) = \overline{H}^{-1}\left(b + \frac{1}{\rho_{M_1|B_1}(b)}\right)$, (1) follows from

$$
\begin{aligned}
\mathrm{E}[X_o|X^{(M)} = x] &= \frac{\mathrm{E}[X_o|X_1 \leq x, Y_1 = x] \int_0^x f_{X_1Y_1}(u,x)du}{f_{X^{(M)}}(x)} + \frac{\mathrm{E}[X_o|X_1 = x, Y_1 \leq x] \int_0^x f_{X_1Y_1}(x,u)du}{f_{X^{(M)}}(x)}\\
&= \frac{\overline{L}(x) \int_0^x f_{X_1Y_1}(u,x)du}{f_{X^{(M)}}(x)} + \frac{\overline{L}(x) \int_0^x f_{X_1Y_1}(x,u)du}{f_{X^{(M)}}(x)},
\end{aligned}
$$

and the last line in the derivation of $\mathrm{E}[X_o|X^{(M)} = x]$ is due to

$$
\begin{aligned}
\mathrm{E}[X_o|X_1 \leq x, Y_1 = x] &= \sum_{m=2}^{M} \frac{1}{M-1} \mathrm{E}[X_o|X_1 \leq x, X_2 \leq x, \cdots, X_m = x, X_{m+1} \leq x, \cdots, X_M \leq x]\\
&= \sum_{m=2}^{M} \frac{1}{M-1} \mathrm{E}[X_o|X_m = x, Y_m \leq x]\\
&= \mathrm{E}[X_o|X_1 = x, Y_1 \leq x] = \overline{L}(x)
\end{aligned}
$$

under the symmetric bidders assumption. (2) follows from changes of variables using the facts that $F_{X^{(M)}}(x) = G_{B^{(M)}}(\beta(x))$, $F_{X_1Y_1}(x_1,y_1) = G_{B_1M_1}(\beta(x_1),\beta(y_1))$ and thus $f_{X_1Y_1}(x_1,y_1) = g_{B_1M_1}(\beta(x_1),\beta(y_1))\beta'(x_1)\beta'(y_1)$.[16]

[16] Notice that $\mathrm{E}[X_o|X^{(M)} \geq x_r^*]\mathrm{E}[\mathbb{1}(X^{(M)} \geq x_r^*)] = \mathrm{E}[X_o|B^{(M)} \geq b_r^*]\mathrm{E}[\mathbb{1}(B^{(M)} \geq b_r^*)]$. When the ex-post common value is known, $\mathrm{E}[X_o|B^{(M)} \geq b_r^*]$ is immediately identified if $\overline{H}(x)$ and $\overline{L}(x)$ are known. One example is the "scaled sale" timber auction in U.S. and Canada, where the quantity of each species of timber extracted from a tract is recorded by an independent agent at the time of harvest, see Athey and Levin (2001).

$\blacksquare$

Proof. (Lemma 3.1) Under Assumption (AS), we can write

$$
\begin{aligned}
\overline{H}(x) &= \mathrm{E}[X_o|X_1 = x, Y_1 = x] \\
&= \mathrm{E}\Big[\frac{1}{M}\sum_{m=1}^{M} X_m|X_1 = x, Y_1 = x\Big] \\
&= \frac{1}{M}\left(x + \mathrm{E}\Big[\sum_{m=2}^{M} X_m|X_1 = x, Y_1 = x\Big]\right) \\
&= \frac{1}{M}\big(x + (M-1)\mathrm{E}\big[X_3|X_1 = x, Y_1 = x\big]\big) \\
&= \frac{1}{M}\left[x + (M-1)\left(\mathrm{E}\big[X_3|X_1 = x, Y_1 = x, Y_1 \neq X_3\big]\frac{M-2}{M-1} + \frac{\mathrm{E}[X_3|X_1 = x, Y_1 = x, Y_1 = X_3]}{M-1}\right)\right] \\
&= \frac{2x}{M} + \frac{(M-2)}{M}\mathrm{E}\big[X_3|X_1 = x, Y_1 = x, Y_1 \neq X_3\big] \\
&= \frac{2x}{M} + \frac{(M-2)}{M}\mathrm{E}\big[X_3|X_1 = x, X_2 = x, X_3 < x, X_4 \leq x, \cdots, X_M \leq x\big] \\
&\overset{(1)}{=} \frac{2x}{M} + \frac{M-2}{M}\int_0^x t d\left(\frac{C_{o,12}(F_o(x), F_o(x), F_o(t), F_o(x), \cdots, F_o(x))}{C_{o,12}(F_o(x), \cdots, F_o(x))}\right) \\
&\overset{(2)}{=} x - \frac{M-2}{M}\frac{\int_0^x C_{o,12}(F_o(x), F_o(x), F_o(t), F_o(x), \cdots, F_o(x))dt}{C_{o,12}(F_o(x), \cdots, F_o(x))},
\end{aligned}
$$

where (1) follows from the fact that for $t \leq x$,

$$
\begin{aligned}
&P(X_3 \leq t|X_1 = x, X_2 = x, X_3 < x, X_4 \leq x, \cdots, X_M \leq x) \\
= &\frac{P(X_3 \leq t, X_4 \leq x, \cdots, X_M \leq x|X_1 = X_2 = x)}{P(X_3 < x, X_4 \leq x, \cdots, X_M \leq x|X_1 = X_2 = x)} \\
= &\frac{C_{o,12}(F_o(x), F_o(x), F_o(t), F_o(x), \cdots, F_o(x))}{C_{o,12}(F_o(x), \cdots, F_o(x))},
\end{aligned}
$$

and (2) is by a change of variable. Similarly, under Assumption (AS),

$$
\begin{aligned}
\overline{L}(x) &= \mathrm{E}\Big[\frac{1}{M}\sum_{m=1}^{M} X_m|X_1 = x, Y_1 \leq x\Big] \\
&= \frac{1}{M}\left(x + \mathrm{E}\Big[\sum_{m=2}^{M} X_m|X_1 = x, Y_1 \leq x\Big]\right) \\
&= \frac{x}{M} + \frac{M-1}{M}\mathrm{E}\big[X_2|X_1 = x, Y_1 \leq x\big] \\
&\overset{(1)}{=} \frac{x}{M} + \frac{M-1}{M}\int_0^x t d\left(\frac{C_{o,1}(F_o(x), F_o(t), F_o(x), \cdots, F_o(x))}{C_{o,1}(F_o(x), \cdots, F_o(x))}\right) \\
&\overset{(2)}{=} x - \frac{M-1}{M}\frac{\int_0^x C_{o,1}(F_o(x), F_o(t), F_o(x), \cdots, F_o(x))dt}{C_{o,1}(F_o(x), \cdots, F_o(x))},
\end{aligned}
$$

where (1) follows from the fact that for $t \leq x$,

$$
\begin{aligned}
P(X_2 \leq t | X_1 = x, X_2 \leq x, \cdots, X_M \leq x) &= \frac{P(X_2 \leq t, X_3 \leq x, \cdots, X_M \leq x | X_1 = x)}{P(X_2 \leq x, \cdots, X_M \leq x | X_1 = x)} \\
&= \frac{C_{o,1}(F_o(x), F_o(t), F_o(x), \cdots, F_o(x))}{C_{o,1}(F_o(x), \cdots, F_o(x))},
\end{aligned}
$$

and (2) is by a change of variable. ■

Proof. (Theorem 3.2) Under Assumption (AS), the restriction on $F_o(x)$ is $R_1(x; F_o, C_o) = 0$, where

$$
\begin{aligned}
R_1(x; F, C_o) &= \overline{H}(x; F, C_o) - Q_{B_1}(F(x)) - \frac{1}{\rho_{M_1|B_1}(Q_{B_1}(F(x)))} \\
&= x - \frac{M-2}{M} \frac{\int_0^x C_{o,12}(F(x), F(x), F(t), F(x), \cdots, F(x)) dt}{C_{o,12}(F(x), \cdots, F(x))} \\
&\quad - Q_{B_1}(F(x)) - \frac{1}{\rho_{M_1|B_1}(Q_{B_1}(F(x)))}.
\end{aligned}
$$

By change of variables $F_o(x) = \tau$, $F_o(t) = s$ and thus $x = Q_o(\tau)$, $t = Q_o(s)$, we get the following equivalent restriction

$$
\begin{aligned}
&Q_o(\tau) - \frac{M-2}{M} \frac{\int_0^\tau C_{o,12}(\tau, \tau, s, \tau, \cdots, \tau) Q_o'(s) ds}{C_{o,12}(\tau, \cdots, \tau)} - Q_{B_1}(\tau) - \frac{1}{\rho_{M_1|B_1}(Q_{B_1}(\tau))} \\
= &Q_o(\tau) - \frac{M-2}{M} \left(Q_o(\tau) - \frac{\int_0^\tau C_{o,123}(\tau, \tau, s, \tau, \cdots, \tau) Q_o(s) ds}{C_{o,12}(\tau, \cdots, \tau)} \right) - Q_{B_1}(\tau) - \frac{1}{\rho_{M_1|B_1}(Q_{B_1}(\tau))} = 0.
\end{aligned}
$$

Upon rearranging the restriction, we obtain

$$
Q_o(\tau) - \int_0^\tau k_{1o}(\tau, s) Q_o(s) ds = \phi_{1o}(\tau),
$$

where

$$
\phi_{1o}(\tau) = \frac{M Q_{B_1}(\tau)}{2} + \frac{M}{2\rho_{M_1|B_1}(Q_{B_1}(\tau))}, \text{ and } k_{1o}(\tau, s) = -\frac{(M-2) C_{o,123}(\tau, \tau, s, \tau, \cdots, \tau)}{2 C_{o,12}(\tau, \cdots, \tau)}.
$$

The restriction above is a linear Volterra integral equation of the second kind. By definition, the true quantile function $Q_o(\tau)$ is solution to the integral equation. Further, under Assumption (CU-1), Theorem 2.1.1. in Burton (2005) or Theorem 3.12 in Kress (1999) can be readily applied and the above Volterra integral equation has a unique solution, thus the true quantile function $Q_o(\tau)$ of bidders' private signals is nonparametrically identified by the above Volterra integral equation. ■

Proof. (Proposition 3.7) Let the seller's own valuation be v_o, in a second-price sealed-bid pure common value auction, the equilibrium bidding function under r as $\beta_r(x) = \overline{H}(x), x \in [x_r^*, \overline{x}]$, $\beta_r(x) < r, x \in [0, x_r^*)$. The seller's profit under reserve price r is

$$\pi_S(r) = v_o \mathbb{1}(X^{(M)} < x_r^*) + r\mathbb{1}(X^{(M)} \geq x_r^*, X^{(M-1)} < x_r^*) + \overline{H}(X^{(M-1)})\mathbb{1}(X^{(M-1)} \geq x_r^*).$$

Then

$$\begin{aligned}
\mathrm{E}[\pi_S(r)] &= v_o \mathrm{E}[\mathbb{1}(X^{(M)} < x_r^*)] + r\mathrm{E}[\mathbb{1}(X^{(M)} \geq x_r^*, X^{(M-1)} < x_r^*)] + \mathrm{E}[\overline{H}(X^{(M-1)})\mathbb{1}(X^{(M-1)} \geq x_r^*)] \\
&= v_o \mathrm{E}[\mathbb{1}(B^{(M)} < \overline{H}(x_r^*))] + \mathrm{E}[\pi_P(r)],
\end{aligned}$$

where $\mathrm{E}[\pi_P(r)] = r\mathrm{E}[\mathbb{1}(B^{(M)} \geq \overline{H}(x_r^*), B^{(M-1)} < \overline{H}(x_r^*))] + \mathrm{E}[B^{(M-1)}\mathbb{1}(B^{(M-1)} \geq \overline{H}(x_r^*))]$. On the other hand, for the bidders' surplus, we have

$$\pi_B(r) = (X_o - r)\mathbb{1}(X^{(M)} \geq x_r^*, X^{(M-1)} < x_r^*) + (X_o - \overline{H}(X^{(M-1)}))\mathbb{1}(X^{(M-1)} \geq x_r^*).$$

Then

$$\begin{aligned}
\mathrm{E}[\pi_B(r)] &= \mathrm{E}[(X_o - r)\mathbb{1}(X^{(M)} \geq x_r^*, X^{(M-1)} < x_r^*)] + \mathrm{E}[(X_o - \overline{H}(X^{(M-1)}))\mathbb{1}(X^{(M-1)} \geq x_r^*)] \\
&= \mathrm{E}[X_o \mathbb{1}(X^{(M)} \geq x_r^*)] - \mathrm{E}[\pi_P(r)] \\
&= \mathrm{E}\left[\mathbb{1}(M_1 \geq \overline{H}(x_r^*))\mathbb{1}(B_1 \leq M_1)\overline{L}(\overline{H}^{-1}(M_1))\right] \\
&\quad + \mathrm{E}\left[\mathbb{1}(B_1 \geq \overline{H}(x_r^*))\mathbb{1}(M_1 \leq B_1)\overline{L}(\overline{H}^{-1}(B_1))\right] - \mathrm{E}[\pi_P(r)],
\end{aligned}$$

where the last step follows similarly as that in the proof of Proposition 2.2, where we need to replace $\beta(\cdot)$ by $\overline{H}(\cdot)$ and to replace $b_r^* = \beta(x_r^*)$ by $\overline{H}(x_r^*)$. ■

Proof. (Theorem 3.9) Under Assumption (AS), the restriction on $F_o(x)$ is $R_3(x; F_o, C_o) = 0$, where

$$\begin{aligned}
R_3(x; F, C_o) = &\rho_{M_1|B_1}(Q_{B_1}(F(x))) \left[\overline{H}(x; F, C_o) - Q_{B_1}(F(x))\right] \\
&+ N\rho_V(Q_{B_1}(F(x))) \left[\overline{L}(x; F, C_o) - Q_{B_1}(F(x))\right].
\end{aligned}$$

By change of variables $F_o(x) = \tau$, $F_o(t) = s$ and thus $x = Q_o(\tau)$, $t = Q_o(s)$, we get the following equivalent restriction

$$
\begin{aligned}
&\rho_{M_1|B_1}(Q_{B_1}(\tau))\left[Q_o(\tau)-\frac{M-2}{M}\frac{\int_0^\tau C_{o,12}(\tau,\tau,s,\tau,\cdots,\tau)Q_o'(s)ds}{C_{o,12}(\tau,\cdots,\tau)}-Q_{B_1}(\tau)\right]\\
&+N\rho_V(Q_{B_1}(\tau))\left[Q_o(\tau)-\frac{M-1}{M}\frac{\int_0^\tau C_{o,1}(\tau,s,\tau,\cdots,\tau)Q_o'(s)ds}{C_{o,1}(\tau,\cdots,\tau)}-Q_{B_1}(\tau)\right]\\
&=\rho_{M_1|B_1}(Q_{B_1}(\tau))\left[\frac{2}{M}Q_o(\tau)+\frac{M-2}{M}\frac{\int_0^\tau C_{o,123}(\tau,\tau,s,\tau,\cdots,\tau)Q_o(s)ds}{C_{o,12}(\tau,\cdots,\tau)}-Q_{B_1}(\tau)\right]\\
&+N\rho_V(Q_{B_1}(\tau))\left[\frac{1}{M}Q_o(\tau)+\frac{M-1}{M}\frac{\int_0^\tau C_{o,12}(\tau,s,\tau,\cdots,\tau)Q_o(s)ds}{C_{o,1}(\tau,\cdots,\tau)}-Q_{B_1}(\tau)\right]\\
&=Q_o(\tau)\left[\frac{2\rho_{M_1|B_1}(Q_{B_1}(\tau))}{M}+\frac{N\rho_V(Q_{B_1}(\tau))}{M}\right]+\int_0^\tau\left(\frac{(M-2)\rho_{M_1|B_1}(Q_{B_1}(\tau))C_{o,123}(\tau,\tau,s,\tau,\cdots,\tau)}{MC_{o,12}(\tau,\cdots,\tau)}\right.\\
&\left.+\frac{N(M-1)\rho_V(Q_{B_1}(\tau))C_{o,12}(\tau,s,\tau,\cdots,\tau)}{MC_{o,1}(\tau,\cdots,\tau)}\right)Q_o(s)ds-\left[\rho_{M_1|B_1}(Q_{B_1}(\tau))+N\rho_V(Q_{B_1}(\tau))\right]Q_{B_1}(\tau)=0.
\end{aligned}
$$

Upon rearranging the restriction, we obtain

$$
Q_o(\tau)-\int_0^\tau k_{3o}(\tau,s)Q_o(s)ds=\phi_{3o}(\tau),
$$

where

$$
\begin{aligned}
\phi_{3o}(\tau)&=\frac{M\left[\rho_{M_1|B_1}(Q_{B_1}(\tau))+N\rho_V(Q_{B_1}(\tau))\right]Q_{B_1}(\tau)}{2\rho_{M_1|B_1}(Q_{B_1}(\tau))+N\rho_V(Q_{B_1}(\tau))},\\
k_{3o}(\tau,s)&=-\frac{(M-2)\rho_{M_1|B_1}(Q_{B_1}(\tau))z_{1o}(\tau,s)}{[2\rho_{M_1|B_1}(Q_{B_1}(\tau))+N\rho_V(Q_{B_1}(\tau))]}-\frac{N(M-1)\rho_V(Q_{B_1}(\tau))z_{2o}(\tau,s)}{[2\rho_{M_1|B_1}(Q_{B_1}(\tau))+N\rho_V(Q_{B_1}(\tau))]}.
\end{aligned}
$$

By definition, the true quantile function $Q_o(\tau)$ is solution to the integral equation. Further, under Assumption (CU-3), Theorem 2.1.1. in Burton (2005) or Theorem 3.12 in Kress (1999) can be readily applied and the above Volterra integral equation has a unique solution, thus the true quantile function $Q_o(\tau)$ of bidders' private signals is nonparametrically identified by the above Volterra integral equation. ■

Proof. (Theorem 4.1) Write

$$
\begin{aligned}
\widehat{Q}(\tau)-Q_o(\tau)&=(I-\widehat{K}_1)^{-1}\widehat{\phi}_1(\tau)-(I-K_{1o})^{-1}\phi_{1o}(\tau)\\
&\overset{(1)}{=}(I-\widehat{K}_1)^{-1}(\widehat{\phi}_1-\phi_{1o})(\tau)+\left[(I-\widehat{K}_1)^{-1}-(I-K_{1o})^{-1}\right]\phi_{1o}(\tau)\\
&\overset{(2)}{=}(I-\widehat{K}_1)^{-1}\left[\widehat{\phi}_1-\phi_{1o}+(\widehat{K}_1-K_{1o})(I-K_{1o})^{-1}\phi_{1o}\right](\tau)\\
&\overset{(3)}{=}(I-\widehat{K}_1)^{-1}\left[(\widehat{\phi}_1-\phi_{1o})+(\widehat{K}_1-K_{1o})Q_o\right](\tau),
\end{aligned}
$$

where (1) follows from the linearity of the operator $(I-\widehat{K}_1)^{-1}$, (2) uses the fact that $A^{-1}-B^{-1}=A^{-1}(B-A)B^{-1}$, (3) follows from the fact that $Q_o(\tau)=(I-K_{1o})^{-1}\phi_{1o}(\tau)$. The claim follows after we show the following two steps.

Step 1: To show $||(I-\widehat{K}_1)^{-1}||_\epsilon - ||(I-K_{1o})^{-1}||_\epsilon = o_P(1)$ for given $\epsilon > 0$, where $||(I-K_1)^{-1}||_\epsilon = \sup_{||\phi||_\epsilon \neq 0} \frac{||(I-K_1)^{-1}\phi||_\epsilon}{||\phi||_\epsilon}$. We first show $||\widehat{K}_1 - K_{1o}||_\epsilon = o_P(1)$, where $||\widehat{K}_1 - K_{1o}||_\epsilon = \sup_{||\varphi||_\epsilon \neq 0} \frac{||(\widehat{K}_1 - K_{1o})\varphi||_\epsilon}{||\varphi||_\epsilon}$ for φ taking values in the space of quantile functions. Here, $||\varphi||_\epsilon = \sup_{t\in[\epsilon,1-\epsilon]} |\varphi(t)|$ and $||(\widehat{K}_1 - K_{1o})\varphi||_\epsilon = \sup_{t\in[\epsilon,1-\epsilon]} |(\widehat{K}_1 - K_{1o})\varphi(t)|$. We have

$$\begin{aligned}
||(\widehat{K}_1 - K_{1o})\varphi||_\epsilon &\equiv \sup_{t\in[\epsilon,1-\epsilon]} \left| \int_0^t [\widehat{k}_1(t,s) - k_{1o}(t,s)]\varphi(s)ds \right| \\
&\leq \sup_{t\in[\epsilon,1-\epsilon]} \int_0^t |\widehat{k}_1(t,s) - k_{1o}(t,s)||\varphi(s)|ds \\
&\leq \sup_{t\in[\epsilon,1-\epsilon]} \int_0^t |\widehat{k}_1(t,s) - k_{1o}(t,s)| \sup_{s\in[\epsilon,1-\epsilon]} |\varphi(s)|ds \\
&\leq ||\varphi||_\epsilon \sup_{t\in[\epsilon,1-\epsilon]} \int_0^1 |\widehat{k}_1(t,s) - k_{1o}(t,s)|ds,
\end{aligned}$$

where the third line follows from the fact that $\varphi(s)$ is increasing. Then it follows that

$$||\widehat{K}_1 - K_{1o}||_\epsilon \leq \frac{M-2}{2} \sup_{t\in[\epsilon,1-\epsilon]} \int_0^1 \left| z_1(t,s;\widehat{\theta}_L) - z_{1o}(t,s) \right| ds = \frac{M-2}{2} w_{1\epsilon}(\widehat{\theta}_L) = o_P(1)$$

under Assumption (RA-1) (i) and the fact that $\widehat{\theta}_L - \theta_o = o_P(1)$. Then

$$\begin{aligned}
&\left| ||(I-\widehat{K}_1)^{-1}||_\epsilon - ||(I-K_{1o})^{-1}||_\epsilon \right| \\
&\overset{(1)}{\leq} ||(I-\widehat{K}_1)^{-1} - (I-K_{1o})^{-1}||_\epsilon \\
&\overset{(2)}{=} ||(I-\widehat{K}_1)^{-1}(\widehat{K}_1 - K_{1o})(I-K_{1o})^{-1}||_\epsilon \\
&\overset{(3)}{\leq} ||(I-\widehat{K}_1)^{-1}||_\epsilon ||\widehat{K}_1 - K_{1o}||_\epsilon ||(I-K_{1o})^{-1}||_\epsilon \\
&\leq \left| ||(I-\widehat{K}_1)^{-1}||_\epsilon - ||(I-K_{1o})^{-1}||_\epsilon \right| ||\widehat{K}_1 - K_{1o}||_\epsilon ||(I-K_{1o})^{-1}||_\epsilon \\
&\quad + ||\widehat{K}_1 - K_{1o}||_\epsilon ||(I-K_{1o})^{-1}||_\epsilon^2,
\end{aligned}$$

where (1) follows from triangle inequality, (2) uses the fact that $A^{-1} - B^{-1} = A^{-1}(B-A)B^{-1}$, (3) follows from $||A\varphi|| \leq ||A|| ||\varphi||$. Let $Z_{1L} = \left| ||(I-\widehat{K}_1)^{-1}||_\epsilon - ||(I-K_{1o})^{-1}||_\epsilon \right|$, $Z_{2L} = ||\widehat{K}_1 - K_{1o}||_\epsilon$, $a_1 = ||(I-K_{1o})^{-1}||_\epsilon$. Upon rearrangement of the above inequality, we get

$Z_{1L}(1-a_1Z_{2L}) \le a_1^2 Z_{2L}$. Given any $\delta > 0$,

$$\begin{aligned}
P(Z_{1L} > \delta) &= P(Z_{1L} > \delta, 1 - a_1Z_{2L} > 0) + P(Z_{1L} > \delta, 1 - a_1Z_{2L} \le 0) \\
&\le P\left(\frac{a_1^2 Z_{2L}}{1-a_1Z_{2L}} \ge Z_{1L} > \delta, 1 - a_1Z_{2L} > 0\right) + P\left(Z_{2L} \ge \frac{1}{a_1}\right) \\
&\le P\left(\frac{a_1^2 Z_{2L}}{1-a_1Z_{2L}} > \delta, 1 - a_1Z_{2L} > 0\right) + P\left(Z_{2L} \ge \frac{1}{a_1}\right) \\
&\le P\left(Z_{2L} > \frac{\delta}{a_1^2 + a_1\delta}\right) + P\left(Z_{2L} \ge \frac{1}{a_1}\right) \to 0, \text{ as } L \to \infty.
\end{aligned}$$

Therefore, $||(I - \widehat{K}_1)^{-1}||_\epsilon - ||(I-K_{1o})^{-1}||_\epsilon = o_P(1)$.

Step 2: To show $||\widehat{\phi}_1 - \phi_{1o} + (\widehat{K}_1 - K_{1o})Q_o||_\epsilon = o_P(1)$. We have

$$\begin{aligned}
&||\widehat{\phi}_1 - \phi_{1o} + (\widehat{K}_1 - K_{1o})Q_o||_\epsilon \\
&\le ||\widehat{\phi}_1 - \phi_{1o}||_\epsilon + ||(\widehat{K}_1 - K_{1o})Q_o||_\epsilon \\
&\le \frac{M}{2} \sup_{t\in[\epsilon,1-\epsilon]} |\widehat{Q}_{B_1}(t) - Q_{B_1}(t)| + \frac{M}{2} \sup_{t\in[\epsilon,1-\epsilon]} \left| \frac{1}{\widehat{\rho}_{M_1|B_1}(\widehat{Q}_{B_1}(t))} - \frac{1}{\rho_{M_1|B_1}(Q_{B_1}(t))} \right| \\
&\quad + \frac{M-2}{2} \sup_{t\in[\epsilon,1-\epsilon]} \int_0^1 |z_1(t,s;\widehat{\theta}_L) - z_{1o}(t,s)| Q_o(s) ds \\
&= o_P(1) + \frac{M}{2} \sup_{t\in[\epsilon,1-\epsilon]} \left| \frac{1}{\widehat{\rho}_{M_1|B_1}(\widehat{Q}_{B_1}(t))} - \frac{1}{\rho_{M_1|B_1}(Q_{B_1}(t))} \right|,
\end{aligned}$$

where the last equality follows from Assumption (RA-1)(ii) and the fact that $\widehat{\theta}_L - \theta_o = o_P(1)$. For the second term, let $[\underline{b}_\epsilon, \overline{b}_\epsilon] \subset [\underline{b}, \overline{b}]$ be a compact strict subset, where $\underline{b}_\epsilon$ is such that $P(\widehat{Q}_{B_1}(\epsilon) \ge \underline{b}_\epsilon) \to 1$ and $\overline{b}_\epsilon$ is such that $P(\widehat{Q}_{B_1}(1-\epsilon) \le \overline{b}_\epsilon) \to 1$. We have

$$\begin{aligned}
&\sup_{t\in[\epsilon,1-\epsilon]} \left| \frac{1}{\widehat{\rho}_{M_1|B_1}(\widehat{Q}_{B_1}(t))} - \frac{1}{\rho_{M_1|B_1}(Q_{B_1}(t))} \right| \\
&\le \sup_{t\in[\epsilon,1-\epsilon]} \left| \frac{1}{\widehat{\rho}_{M_1|B_1}(\widehat{Q}_{B_1}(t))} - \frac{1}{\rho_{M_1|B_1}(\widehat{Q}_{B_1}(t))} \right| + \sup_{t\in[\epsilon,1-\epsilon]} \left| \frac{1}{\rho_{M_1|B_1}(\widehat{Q}_{B_1}(t))} - \frac{1}{\rho_{M_1|B_1}(Q_{B_1}(t))} \right| \\
&\equiv Z_{3L} + Z_{4L}.
\end{aligned}$$

For Z_{3L}, let $Z_{3L}^* = \sup_{b\in[\underline{b}_\epsilon, \overline{b}_\epsilon]} \left| \frac{1}{\widehat{\rho}_{M_1|B_1}(b)} - \frac{1}{\rho_{M_1|B_1}(b)} \right|$, then for any $\delta > 0$,

$$\begin{aligned}
P(Z_{3L} \ge \delta) &\le P(Z_{3L} \ge \delta, Z_{3L}^* \ge Z_{3L}) + P(Z_{3L} \ge \delta, Z_{3L}^* < Z_{3L}) \\
&\le P(Z_{3L}^* \ge \delta, Z_{3L}^* \ge Z_{3L}) + P(Z_{3L}^* < Z_{3L}) \\
&\le P(Z_{3L}^* \ge \delta) + P(Z_{3L}^* < Z_{3L}) \to 0,
\end{aligned}$$

where $P(Z_{3L}^* < Z_{3L}) \to 0$ follows from

$$\begin{aligned} P(Z_{3L}^* \geq Z_{3L}) &\geq P\left(\widehat{Q}_{B_1}(t) \in [\underline{b}_\epsilon, \overline{b}_\epsilon] \text{ for all } t \in [\epsilon, 1-\epsilon]\right) \\ &= P\left(\inf_{t\in[\epsilon,1-\epsilon]} \widehat{Q}_{B_1}(t) \geq \underline{b}_\epsilon, \sup_{t\in[\epsilon,1-\epsilon]} \widehat{Q}_{B_1}(t) \leq \overline{b}_\epsilon\right) \\ &= P\left(\widehat{Q}_{B_1}(\epsilon) \geq \underline{b}_\epsilon, \widehat{Q}_{B_1}(1-\epsilon) \leq \overline{b}_\epsilon\right) \\ &\geq P\left(\widehat{Q}_{B_1}(\epsilon) \geq \underline{b}_\epsilon\right) - P\left(\widehat{Q}_{B_1}(1-\epsilon) > \overline{b}_\epsilon\right) \to 1. \end{aligned}$$

We also need to show $P(Z_{3L}^* \geq \delta) \to 0$. We have $\frac{1}{\widehat{\rho}_{M_1|B_1}(b)} = \frac{\widehat{G}_{M_1\times B_1}(b)}{\widehat{g}_{M_1B_1}(b)}$, $\frac{1}{\rho_{M_1|B_1}(b)} = \frac{G_{M_1\times B_1}(b)}{g_{M_1B_1}(b)}$, where $G_{M_1\times B_1}(b) = \partial G_{M_1B_1}(m_1, b_1)/\partial b_1|_{m_1=b_1=b}$. Li, Perrigne, and Vuong (2002) show that $\sup_{b\in[\underline{b}_\epsilon,\overline{b}_\epsilon]} |\widehat{G}_{M_1\times B_1}(b) - G_{M_1\times B_1}(b)| = o_P(1)$, and $\sup_{b\in[\underline{b}_\epsilon,\overline{b}_\epsilon]} |\widehat{g}_{M_1B_1}(b) - g_{M_1B_1}(b)| = o_P(1)$. Then

$$\begin{aligned} &\sup_{b\in[\underline{b}_\epsilon,\overline{b}_\epsilon]} \left|\frac{1}{\widehat{\rho}_{M_1|B_1}(b)} - \frac{1}{\rho_{M_1|B_1}(b)}\right| \\ &= \sup_{b\in[\underline{b}_\epsilon,\overline{b}_\epsilon]} \left|\frac{\widehat{G}_{M_1\times B_1}(b)}{\widehat{g}_{M_1B_1}(b)} - \frac{\widehat{G}_{M_1\times B_1}(b)}{g_{M_1B_1}(b)}\right| + \sup_{b\in[\underline{b}_\epsilon,\overline{b}_\epsilon]} \left|\frac{\widehat{G}_{M_1\times B_1}(b)}{g_{M_1B_1}(b)} - \frac{G_{M_1\times B_1}(b)}{g_{M_1B_1}(b)}\right| \\ &\leq \sup_{b\in[\underline{b}_\epsilon,\overline{b}_\epsilon]} |\widehat{G}_{M_1\times B_1}(b)| \sup_{b\in[\underline{b}_\epsilon,\overline{b}_\epsilon]} \left|\frac{1}{\overline{g}^2_{M_1B_1}(b)}\right| \sup_{b\in[\underline{b}_\epsilon,\overline{b}_\epsilon]} |\widehat{g}_{M_1B_1}(b) - g_{M_1B_1}(b)| \\ &\quad + \sup_{b\in[\underline{b}_\epsilon,\overline{b}_\epsilon]} \left|\frac{1}{g_{M_1B_1}(b)}\right| \sup_{b\in[\underline{b}_\epsilon,\overline{b}_\epsilon]} \left|\widehat{G}_{M_1\times B_1}(b) - G_{M_1\times B_1}(b)\right| \\ &\overset{(1)}{\leq} \sup_{b\in[\underline{b}_\epsilon,\overline{b}_\epsilon]} |\widehat{G}_{M_1\times B_1}(b) - G_{M_1\times B_1}(b)| \sup_{b\in[\underline{b}_\epsilon,\overline{b}_\epsilon]} \left|\frac{1}{\overline{g}^2_{M_1B_1}(b)}\right| \sup_{b\in[\underline{b}_\epsilon,\overline{b}_\epsilon]} |\widehat{g}_{M_1B_1}(b) - g_{M_1B_1}(b)| \\ &\quad + \sup_{b\in[\underline{b}_\epsilon,\overline{b}_\epsilon]} |G_{M_1\times B_1}(b)| \sup_{b\in[\underline{b}_\epsilon,\overline{b}_\epsilon]} \left|\frac{1}{\overline{g}^2_{M_1B_1}(b)}\right| \sup_{b\in[\underline{b}_\epsilon,\overline{b}_\epsilon]} |\widehat{g}_{M_1B_1}(b) - g_{M_1B_1}(b)| \\ &\quad + \sup_{b\in[\underline{b}_\epsilon,\overline{b}_\epsilon]} \left|\frac{1}{g_{M_1B_1}(b)}\right| \sup_{b\in[\underline{b}_\epsilon,\overline{b}_\epsilon]} \left|\widehat{G}_{M_1\times B_1}(b) - G_{M_1\times B_1}(b)\right| \overset{(2)}{=} o_P(1), \end{aligned}$$

where $\overline{g}_{M_1B_1}(b)$ is between $\widehat{g}_{M_1B_1}(b)$ and $g_{M_1B_1}(b)$, (1) follows from triangle inequality, (2) follows from Assumptions (UB) (ii) and (iii). In sum, we have $Z_{3L} = o_P(1)$. On the other hand,

$$Z_{4L} = \sup_{t\in[\epsilon,1-\epsilon]} \left|\frac{1}{\rho_{M_1|B_1}(\widehat{Q}_{B_1}(t))} - \frac{1}{\rho_{M_1|B_1}(Q_{B_1}(t))}\right| = o_P(1)$$

due to the uniform continuity of $\frac{1}{\rho_{M_1|B_1}(b)}$ on $[\underline{b}, \overline{b}]$ under Assumption (UB) (i) and the fact that $\sup_{t\in[\epsilon,1-\epsilon]} \left|\widehat{Q}_{B_1}(t) - Q_{B_1}(t)\right| = o_P(1)$.

Now putting Step 1 and Step 2 together, we have

$$\begin{aligned}
\big|\big|\widehat{Q}-Q_o\big|\big|_\epsilon &\le \big|\big|(I-\widehat{K}_1)^{-1}\big|\big|_\epsilon \big|\big|(\widehat{\phi}_1-\phi_{1o})+(\widehat{K}_1-K_{1o})Q_o\big|\big|_\epsilon \\
&\le \left(||(I-\widehat{K}_1)^{-1}||_\epsilon - \big|\big|(I-K_{1o})^{-1}\big|\big|_\epsilon\right)\big|\big|(\widehat{\phi}_1-\phi_{1o})+(\widehat{K}_1-K_{1o})Q_o\big|\big|_\epsilon \\
&+ \big|\big|(I-K_{1o})^{-1}\big|\big|_\epsilon \big|\big|(\widehat{\phi}_1-\phi_{1o})+(\widehat{K}_1-K_{1o})Q_o\big|\big|_\epsilon \\
&= o_P(1)o_P(1)+o(1)o_P(1)+O(1)o_P(1)=o_P(1),
\end{aligned}$$

where the last step follows from Assumption (RA-1)(iii). ■

Proof. (Theorem 4.2)

Similar to the proof of Theorem 4.1, write

$$\widehat{Q}(\tau)-Q_o(\tau)=(I-\widehat{K}_2)^{-1}\left[(\widehat{\phi}_2-\phi_{2o})+(\widehat{K}_2-K_{2o})Q_o\right](\tau),$$

where

$$\begin{aligned}
\widehat{\phi}_2(\tau)-\phi_{2o}(\tau) &= \frac{M}{2}\left[\widehat{Q}_{B_1}(t)-Q_{B_1}(t)\right], \\
(\widehat{K}_2-K_{2o})Q_o(\tau) &= -\frac{M-2}{2}\int_0^\tau \left[z_1(\tau,s;\widehat{\theta}_L)-z_{1o}(\tau,s)\right]Q_o(s)ds.
\end{aligned}$$

The claim follows from the following two steps.

Step 1: To show that $\big|\big|\widehat{\phi}_2-\phi_{2o}+(\widehat{K}_2-K_{2o})Q_o\big|\big|_\infty=o_P(1)$. We have

$$\begin{aligned}
&\left|\left|\widehat{\phi}_2-\phi_{2o}+(\widehat{K}_2-K_{2o})Q_o\right|\right|_\infty \\
\le &\left|\left|\widehat{\phi}_2-\phi_{2o}\right|\right|_\infty+\left|\left|(\widehat{K}_2-K_{2o})Q_o\right|\right|_\infty \\
\le &\frac{M}{2}\sup_{t\in[0,1]}\left|\widehat{Q}_{B_1}(t)-Q_{B_1}(t)\right|+\frac{M-2}{2}\sup_{t\in[0,1]}\int_0^1\left|z_1(t,s;\widehat{\theta}_L)-z_{1o}(t,s)\right|Q_o(s)ds \\
= &o_P(1),
\end{aligned}$$

where the last equality follows from Assumption (RA-1)(ii) and the continuity of $Q_{B_1}(\tau)$ on $[0,1]$.

Step 2: To show that $||(I-\widehat{K}_2)^{-1}||_{op}-||(I-K_{2o})^{-1}||_{op}=o_P(1)$. Following similar argument as in the proof of Theorem 4.1, we can show that under Assumption (RA-2)

$$\left|\left|\widehat{K}_2-K_{2o}\right|\right|_{op}\le\frac{M-2}{2}\sup_{\tau\in[0,1]}\int_0^\tau\left|z_1(\tau,s;\widehat{\theta}_L)-z_1(\tau,s)\right|ds=o_P(1).$$

Also, we can show that $Z_{5L}(1-a_2Z_{6L})\le a_2^2Z_{6L}$, where $Z_{5L}=\Big|||(I-\widehat{K}_2)^{-1}||_{op}-||(I-K_{2o})^{-1}||_{op}\Big|$, $Z_{6L}=||\widehat{K}_2-K_{2o}||_{op}$, $a_2=||(I-K_{2o})^{-1}||_{op}$. $a_2<\infty$ since under Assumption (CU-2), $I-K_{2o}$

is invertible and its inverse $(I - K_{2o})^{-1}$ is continuous thus bounded. Given any $\delta > 0$, similar argument as in the proof of Theorem 4.1 leads to $P(Z_{5L} > \delta) \to 0$. Now putting Step 1 and Step 2 together, we have

$$\begin{aligned}
\left|\left|\widehat{Q} - Q_o\right|\right|_\infty &\leq \left|\left|(I - \widehat{K}_2)^{-1}\right|\right|_{op} \left|\left|(\widehat{\phi}_2 - \phi_{2o}) + (\widehat{K}_2 - K_{2o})Q_o\right|\right|_\infty \\
&\leq \left(||(I - \widehat{K}_2)^{-1}||_{op} - ||(I - K_{2o})^{-1}||_{op}\right) \left|\left|(\widehat{\phi}_2 - \phi_{2o}) + (\widehat{K}_2 - K_{2o})Q_o\right|\right|_\infty \\
&+ ||(I - K_{2o})^{-1}||_{op} \left|\left|(\widehat{\phi}_2 - \phi_{2o}) + (\widehat{K}_2 - K_{2o})Q_o\right|\right|_\infty \\
&= o_P(1)o_P(1) + o_P(1) = o_P(1).
\end{aligned}$$

■

References

[1] Aradilla-López, A., A. Gandhi, and D. Quint (2013), "Identification and Inference in Ascending Auctions with Correlated Private Values," *Econometrica* 81, 489-534.

[2] Armstrong, T. (2013), "Bounds in Auctions with Unobserved Heterogeneity," *Quantitative Economics* 4, 377-415.

[3] Athey, S. and P. Haile (2007), "Nonparametric Approaches to Auctions," *In: Heckman, J.J., Leamer, E. (Eds.), Handbook of Econometrics,* Vol.6. North Holland, Amsterdam, 3849-3959.

[4] Athey, S. and J. Levin (2001), "Information and Competition in US Forest Service Timber Auctions," *Journal of Political Economy* 109, 375-417.

[5] Bajari, P. and A. Hortacsu (2003), "The Winner's Curse, Reserve Prices and Endogenous Entry: Empirical Insights from eBay Auctions," *RAND Journal of Economics* 3, 329-355.

[6] Burton, T. (2005), "Volterra Integral and Differential Equations, Volume 202 of Mathematics in Science and Engineering," 2nd Edition. Elsevier B. V., Amsterdam.

[7] Campo, S., E. Guerre, I. Perrigne, and Q. Vuong (2011), "Semiparametric Estimation of First-Price Auctions with Risk Averse Bidders," *Review of Economic Studies* 78, 112-147.

[8] Carrasco, M., J.-P. Florens, and E. Renault (2007), "Linear Inverse Problems in Structural Econometrics Estimation Based on Spectral Decomposition and Regularization,"

In: Heckman, J.J., Leamer, E. (Eds.), Handbook of Econometrics, Vol.6. North Holland, Amsterdam, 5633-5751.

[9] Chen, X. (2007), "Large Sample Sieve Estimation of Semi-Nonparametric Models," *In: Heckman, J.J., Leamer, E. (Eds.), Handbook of Econometrics,* Vol.6. North Holland, Amsterdam, 5549-5632.

[10] Chen, X., Y. Fan and V. Tsyrennikov (2006), "Efficient Estimation of Semiparametric Multivariate Copula Models," *Journal of the American Statistical Association* 101, 1228-1240.

[11] Chen, Y.-H. (2010), "Semiparametric Marginal Regression Analysis for Dependent Competing Risks under An Assumed Copula," *Journal of Royal Statistical Society B* 72, 235-251.

[12] Darolles, S., Y. Fan, J.-P. FLorens, and E. Renault (2011), "Nonparametric Instrumental Regression," *Econometrica* 79, 1541-1565.

[13] Fan, Y. and X. Liu (2013), "Identification and Inference in Censored Parametric Quantile Regression: Sensitivity to Independent Censoring," *Working paper*, University of Washington.

[14] Fan, Y., T. Li, and M. Pesendorfer (2015), "New Local Polynomial Quantile Estimation of First-Price Auctions," *Working paper.*

[15] Fan, Y., M. He, and T. Li (2015), "A Sensitivity Analysis in the Affiliated Private Value Auction Models with Incomplete Sets of Bids," *Working paper.*

[16] Février, P. (2008), "Nonparametric Identification and Estimation of A Class of Common Value Auction Models," *Journal of Applied Econometrics* 23, 949-964.

[17] Florens, J.-P. (2003), "Inverse Problems in Structural Econometrics: The Example of Instrumental Variables," *In: Dewatripont, M., Hansen, L.P., Turnovsky, S.J. (Eds.), Advances in Economics and Econometrics,* Vol.2. Cambridge University Press, Cambridge, 284-311.

[18] Genest, C., K. Ghoudi, and L.-P. Rivest (1995), "A Semiparametric Estimation Procedure of Dependence Parameters in Multivariate Families of Distributions," *Biometrika* 82, 543-552.

[19] Genest, C., J. Nešlehová, and N. Ghorbal (2011), "Estimators based on kendall's tau in multivariate copula models," *Australian & New Zealand Journal of Statistics*, 53, 157-177.

[20] Gentry, M., and T. Li (2014), "Identification in Auctions with Selective Entry," *Econometrica*, 82, 315-344.

[21] Gentry, M., T. Li, and J. Lu (2015), "Identification and Estimation in First-Price Auctions with Risk-Averse Bidders and Selective Entry," *Working paper.*

[22] Guerre, E., I. Perrigne, and Q. Vuong (2000), "Optimal Nonparametric Estimation of First-Price Auctions," *Econometrica* 68, 525-574.

[23] Guerre, E., I. Perrigne, and Q. Vuong (2009), "Nonparametric Identification of Risk Aversion in First-Price Auctions under Exclusion Restrictions," *Econometrica* 77, 1193-1227.

[24] Goeree, J.K. and T. Offerman (2002), "Efficiency in Auctions with Private and Common Values," *American Economic Review* 92, 625-643.

[25] Goeree, J.K. and T. Offerman (2003), "Competitive Bidding in Auctions with Private and Common Values," *The Economic Journal* 113, 598-613.

[26] Haile, P. and E. Tamer (2003), "Inference with An Incomplete Model of English Auctions," *Journal of Political Economy* 111, 1-52.

[27] Haile, P., H. Hong, and M. Shum (2006), "Nonparametric Tests for Common Values In First-Price Sealed-Bid Auctions," *Working Paper.*

[28] Henderson, D.J., R.J. Carroll, and Q. Li (2008), "Nonparametric Estimation and Testing of Fixed Effects Panel Data Models," *Journal of Econometrics* 144, 257-275.

[29] Hendricks, K., J. Pinkse, and R. Porter (2003), "Empirical Implications of Equlibrium Bidding in First-Price, Symmetric, Common Value Auctions," *Review of Economics Studies* 70, 115-145.

[30] Hendricks, K., R. Porter, and G. Tan (2008), "Bidding Rings and the Winner's Curse," *RAND Journal of Economics* 39, 1018-1041.

[31] Hendricks, K. and R. Porter (2007), "An Empirical Perspective on Auctions," *Handbook of Industrial Organization,* Vol.3. Elsevier.

[32] Hickman, B., T. Hubbard, and Y. Sağlam (2012), "Structural Econometric Methods in Auctions: A Guide to the Literature," *Journal of Econometric Methods* 1, 67-106.

[33] Hong, H. and M. Shum (2002), "Increasing Competition and the Winner's Curse: Evidence from Procurement," *Review of Economic Studies* 69, 871-898.

[34] Huang, X. and N. Zhang (2008), "Regression Survival Analysis with An assumed Copula for Dependent Censoring: A Sensitivity Analysis Approach," *Biometrics* 64, 1090-1099.

[35] Hubbard, T., T. Li, and H.J. Paarsch (2012), "Semiparametric Estimation in Models of First-Price, Sealed-Bid Auctions with Affiliation," *Journal of Econometrics* 168, 4-16.

[36] Klemperer, P.D. (1998), "Auctions with Almost Common Values," *European Economic Review* 42, 757-769.

[37] Komarova, T. (2013), "A New Approach to Identifying Generalized Competing Risks Models with Application to Second-Price Auctions," *Quantitative Economics* 4, 269-328.

[38] Krasnokutskaya, E. (2011), "Identification and Estimation in Procurement Auctions under Unobserved Auction Heterogeneity," *Review of Economic Studies* 28, 293-327.

[39] Kress, R. (1999), "Linear Integral Equations," Springer, New York.

[40] Krishna, V. (2010), "Auction Theory," 2nd Edition, Academic Press.

[41] Laffont, J-J. and Q. Vuong (1996), "Structural Analysis of Auction Data," *American Economic Review: Papers and Proceedings* 86, 414-420.

[42] Li, T., I. Perrigne, and Q. Vuong (2000), "Conditionally Independent Private Information in OCS Wildcat Auctions," *Journal of Econometrics* 98, 129-161.

[43] Li, T., I. Perrigne, and Q. Vuong (2002), "Structural Estimation of the Affiliated Private Value Auction Model," *RAND Journal of Economics* 33, 171-193.

[44] Li, T., I. Perrigne, and Q. Vuong (2003), "Semiparametric Estimation of the Optimal Reserve Price in First-Price Auctions," *Journal of Business & Economic Statistics* 21, 53-64.

[45] Li, T. (2005), "Econometrics of First-Price Auctions with Entry and Binding Reserve Prices," *Journal of Econometrics* 126, 173-200.

[46] Li, T. and X. Zheng (2009), "Entry and Competition Effects in First-Price Auctions: 'Theory and Evidence from Procurement Auctions, " *Review of Econmics Studies* 76, 1397-1429.

[47] Mammen, E., B. Støve, and D. Tjøstheim (2009), "Nonparametric Additive Models for Panels of Time series," *Econometric Theory* 25, 442-481.

[48] Marmer, V. and A. Shneyerov (2012), "Quantile-Based Nonparametric Inference for First-Price Auctions," *Journal of Econometrics* 167, 345-357.

[49] McAfee, R. and D. Vincent (1992), "Updating the Reserve Price in Common Value Models," *American Economic Review, Papers and Proceedings* 82, 512-518.

[50] McNeil, A. J. and J. Neslehova (2009), "Multivariate Archimedean Copulas, d-Monotone Functions and l_1-Norm Symmetric Distributions," *Annals of Statistics* 37, 3059-3097.

[51] Milgrom, P. and R. Weber (1982), "A Theory of Auctions and Competitive Bidding," *Econometrica* 50, 1089-1122.

[52] Nelsen, R. (2006), "An Introduction to Copulas," 2nd Edition, Springer, New York.

[53] Nielsen, J.P. and S. Sperlich (2005), "Smooth Backfitting in Practice," *Journal of the Royal Statistical Society, Series B* 67, 43-61.

[54] Oakes, D. (1994), "Multivariate Survival Distributions," *Journal of Nonparametric Statistics* 3, 343-354.

[55] Paarsch, H.J. (1992), "Deciding Between the Common and Private Value Paradigms in Empirical Models of Auctions," *Journal of Econometrics* 51, 191-215.

[56] Paarsch, H.J. and H. Hong (2006), "An Introduction to the Structural Econometrics of Auction Data,"The MIT Press, Cambridge, MA.

[57] Porter, R. (1995), "The Role of Information in U.S. Offshore Oil and Gas Lease Auction," *Econometrica*, 63, 1-27.

[58] Riley, J. and W. Samuelson (1981), "Optimal Auctions," *American Economic Review* 71, 381-392.

[59] Somaini, P. (2015), "Identification in Auction Models with Interdependent Costs," *Working Paper.*

[60] Su, L. and X. Lu (2013), "Nonparametric Dynamic Panel Data Models: Kernel Estimation and Specification Testing," *Journal of Econometrics* 176, 112-133.

[61] Tan, X. and Y. Xing (2011), "Auctions with Both Common-value and Private-value bidders," *Economics Letters* 111, 99-103.

[62] Tang, X. (2011), "Bounds on Revenue Distributions in Counterfactural Auctions with Reserve Prices," *RAND Journal of Economics* 42, 175-203.

[63] Wilson, R. (1977), "A Bidding Model of Perfect Competition," *Review of Economic Studies* 44, 511-518.

[64] Zheng, M. and J.P. Klein (1995), "Estimates of Marginal Survival for Dependent Competing Risks Based on An Assumed Copula," *Biometrika* 82, 127-138.