Analysis of Demographic Trends on International Interdependence

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Abstract

This paper develops a two-country overlapping generations neoclassical growth model including a realistic demographic structure for the purpose of analyzing the impact of country-level asymmetries in demographic and structural characteristics on cross-country interdependence. I develop two modeling frameworks, with and without a pay-as-you-go social security system and a mandatory retirement age. I find that an increase in the relative life expectancy of a population will produce a positive per-capita net foreign asset position. This is generated by the fact that the country will be comprised of individuals who save relatively more in order to smooth consumption over their extended lifetimes. Additionally, changes in the population growth rate will exert significant pressures on the per-capita dynamics of the net foreign asset position. A relative increase in the population growth rate will exacerbate a decline in the modeled net foreign asset position. Furthermore, I demonstrate how cross-country differences in the rate of time preference will augment the net foreign asset position generated by the demographic transition. Lastly, I calculate a measure of inequality associated with the accumulation of assets over the life cycle. I find that a fall in the population growth rate causes a significant decrease in wealth inequality.

Keywords: Demographic transition, Net foreign assets, International capital flows

JEL Classification: D91, F21, F41, H55, J11

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1. Introduction

Over the last two decades foreign holdings of US assets have significantly surpassed American claims on the rest of the world's, generating a current account deficit and a negative net foreign asset (NFA) position. From 1980 to 2007, the US real NFA position has dropped by a substantial 530 percent.[1] Conversely, some of the United States' largest trading partners, including Canada, China, Germany, and Japan, have all experienced significant increases in their NFA position, with Japan experiencing the largest increase of over 7,000 percent.

Over this same period, a significant demographic transition, centering on a decrease in mortality and a fall in fertility rates, has occurred in many regions across the world. For instance, from 1980 to 2010, the European Union experienced an increase in life expectancy by 6.8 years and a fall in the population growth rate by 46 percent.[2] For many countries this transition is a cause for concern. Countries, such as Japan and Germany, are experiencing a rapid aging of their population, straining social programs dependent upon a large taxable employment base.

This study demonstrates how demographic asymmetries across countries, including differences in age distributions and population growth rates, have a substantial impact on current account balances and NFA positions and can explain a significant portion of their observed trends. I augment the demographic analysis by incorporating observed structural changes, including the evolution of country-specific total factor productivity (TFP), capital share production parameters, and social security policies. Furthermore, I include a brief examination of the impact of an international asymmetry in the pure rate of time preference. Finally, the "natural rate of inequality", as defined by Mierau and Turnovsky (2014a), is measured in order to examine how the associated demographic and structural changes impact the distribution of wealth across the life cycle.

To analyze these changes, I develop a two-country neoclassical growth model including a realistic demographic structure. The two-country model framework is composed of the United States and a population-weighted average of trading partners including Canada, France, Germany, Japan, and the United Kingdom effectively making the structure a one-country one-region model. China is excluded from the list due to the unreliable nature of its demographic data.[3] Each economy is populated by an overlapping generations (OLG) of individuals that

[1] NFA data retrieved from Lane and Milesi-Ferretti (2007).
[2] US Census: International Database.
[3] Gu and Cai (2009) state that the underreporting rate in some provinces reached 37.3 percent of newborns.

differ with respect to their age and their employment status. Two frameworks are utilized in order to decompose the separate demographic and structural influences on the international NFA position. The baseline model consists of agents born into the workforce and employed for their entire finite lifetime. I extend this analysis by including an additional framework that incorporates an exogenous retirement age and a pay-as-you-go social security policy. The social security system is funded through a wage income tax and pays a benefit to the retirees proportional to the per-capita wage level. For tractability, all agents are born as workers, do not reproduce, and do not receive bequests.

Through the use of an empirically estimated survival function the observed life tables specifying the probability of death per age are accurately modeled. The parameterized demographic survival functions are estimated through the use of nonlinear least squares to match country mortality data using the realistic yet tractable function developed by Boucekkine et al. (2002). This allows for the introduction of a credible age-varying probability of death that influences the saving behavior of the modeled economic agent. The international transition occurring during the period from 1980 to 2010 is the focus of the analysis. This time frame is long enough to generate significant demographic change. Because of the thorough integration of both the financial and goods markets of the chosen countries, cross-country interdependence is modeled through the use of perfect capital markets and a single traded consumption good.

Due to the complicated nature of the dynamic system, numerical simulations are used to analyze the impact of the 30-year demographic and structural transition. I first analyze the impact of the fall in mortality rates associated with an increase in life expectancy with and without the observed change in the population growth rate. Structural changes are then included in the analysis along with the full demographic transition. I then analyze the impact of the inclusion of a retirement age and social security system. The retirement age and the gross replacement rate are set exogenously, thereby allowing the tax rate to be endogenously determined through the interaction of the labor force participation rate and the per-capita wage level. Finally, for the baseline model, the Gini coefficient is estimated measuring the wealth inequality of the countries. The coefficient is calculated for each steady state, reflecting the equilibrium age distribution of assets before and after each shock.

Using these frameworks I obtain the following results. The per-capita NFA position is directly linked to the relative life expectancies of the population and the fertility rates of the countries. Regardless of the inclusion of a retirement period, the country that experiences a relatively higher life expectancy will save more and accumulate more wealth. Due to the increase in the relative saving rate, a positive NFA position is generated for the country. A decrease in the population growth rate will have a similar effect. The decreased arrival of newborns into the economy will increase per-capita wealth leading the country to become an international lender. Furthermore, the country that experiences a relatively higher rate of time preference will produce a negative NFA position due to their inclination for current consumption. Over the 30-year period the average life expectancy of the region (Canada, France, Japan, Germany, and the UK) remained higher and increased by more than that of the US. Additionally, during the same time frame, the US experienced a relatively larger population growth rate and a smaller decline. This overall effect led the US to generate a negative and declining NFA position.

Including a social security system and matching the observed gross benefit replacement rates for the period drives the American NFA position to an initially positive value. The preliminary positive position is a result of the relatively high benefit rate for the trading region, limiting the necessity for the agents to save for retirement. However, the massive reduction in the region's replacement rate over the time interval reverses this position and the long run American NFA value is driven negative once again.

The remainder of the paper is structured as follows. Section 2 discusses relevant literature. Section 3 explains recent demographic and employment trends experienced by the sampled countries. Section 4 and 5 lay out the two analytical frameworks. Sections 6 and 7 describe the numerical simulation and Section 8 concludes the paper.

2. Related Literature

While the inclusion of realistic demographic structures in international models is uncommon, this study is not the first to utilize them. Attanasio and Violante (2000) study the impact of a demographic transition on factor returns and cohort welfare levels as countries move from an autarkic state to a perfect open capital market. They find that the liberalization of the capital market will exacerbate the flows of capital associated with country-specific differentials in the rate of return due to asymmetric demographic structures across countries. While their analysis is close to mine, they calibrate the modeled regions to resemble the US and Europe as one region and Latin America as another. Additionally, I focus on the international setting post-liberalization. I model how recent developments in demographic and structural trends impact the flows of capital after the rates of return have been equalized.

Feroli (2003) develops an open economy model calibrated to match the G-7 nations. Using a similar model with different countries he is able to match certain trends in the NFA position for the US and Japan. The main difference in his modeling technique is his estimated demographic framework and the regional structure. The modeled demographics in his study are calibrated to match the observed and predicted population counts for 5-year intervals for the years 1950 to 2050 from the US Census Bureau's International Database. Agents in his model live with certainty for 12 five-year periods from age 20 to 80.

Ghironi (2006) develops a two-country overlapping generations model to study the role of NFA's in the transmission of productivity shocks. He highlights the mechanism by which the generational structure allows for the determinacy of the asset position in an incomplete markets setup. He exhibits the negative effect of certain assumptions associated with the removal of current account dynamics. His model is similar to mine, yet his focus is very different. His analysis focuses on the importance of using the overlapping generations framework in order to find a stationary steady state for the NFA position. Additionally, in order to increase tractability, he removes realistic demographic structures instead relying upon infinitely lived agents, more closely resembling the Blanchard (1985) model.

Ferrero (2010) uses a multi-country model to decompose the US trade balance into demographic and productivity factors. He simplifies the demographic structure by disentangling the survival probability from the age of the agent. However, he does include a stochastic retirement age but eschews from including any social security system. With this framework he is able to generate a strikingly realistic declining trade balance for the US. Additionally, he is able to mimic the decline in the international real interest rate produced by the increase in the world supply of savings.

Lastly, Backus et al. (2014) develop a multi-country open economy model looking at the impact of simulated demographic trends on the capital flows across countries. They calibrate the model to match the following countries: China, Germany, Japan, and the US. Through the modeling of an approximated demographic transition they are able to show that cross-country demographic differences have a significant impact on international capital flows. Unlike their analysis, I focus on differentiating the effects of the specific aspects of the demographic and structural trends in order to observe their individual impact on capital flows. I am also able to generate a more realistic negative NFA position and transition for the US, something that is not possible in their multi-country model.

3. International Demographic Trends

This section briefly describes the significant population and employment trends experienced by the sampled countries in a greater level of detail. Through medical and lifestyle revolutions during the period from 1980 to 2010 a majority of countries experienced an increase in life expectancy. For the included countries the average life expectancy at birth in 1980 and 2010 was 74.2 and 80.7, respectively. This is an increase in life expectancy of 1.3 years per five years on average. The increase in life expectancy is a function of the decrease in the age-specific probability of death over the life cycle. This trend has been styled the "rectangularization of the survival function".[4] It refers to the survival function becoming a more box-like or rectangular shape as shown by Figure 1 for the US from 1980 to 2010.

The population growth rate utilized in this study is calculated by the World Bank as including all residents of the country regardless of legal status. The fertility rate is measured by the number of children that would be born to a specific woman living to the end of her childbearing years. Given that the fertility rates for the included countries remained largely stagnant and, with the exception of France and the UK, the annual population growth rates declined, this has led many countries to reach population growth rates significantly below their replacement rate, leading to the general aging of the population.

The aging of the population has become a serious challenge for the funding of national pensions schemes for many countries. On average, as shown by Table 8, during the same time period the age dependency ratio, as measured by the percentage of people aged 64 and older to the working age population, has increased from 18.9 to 26.3 percent. Due to the fact that the effective retirement age has stayed fairly constant, with the exception of France where it has decreased by 4.0 years, the gross pension replacement rates have fallen substantially for all the sampled countries excluding Canada. With the exception of Canada, the average percent change in gross pension benefits amounted to a 24 percent decrease for the countries included in the trading region as opposed to a decrease by 12 percent for the US during the 30-year period.

Much of the reduction in retirement benefits is due to the trend of falling labor force participation rates. The labor force participation rate is defined as the fraction of the population that is employed in a country. As shown by Table 10, every country in my sample has experienced a decrease in the participation rate in the last twenty years, with the UK experiencing the largest decrease of 8.2 percent and the overall average decrease amounting to 6.4 percent. This change in the labor force will alter the tax base associated with the national

[4] See Rossi, I.A., V. Rousson, and F. Paccaud (2013).

pension scheme. A falling participation rate requires an increase in the social security tax in order to maintain constant benefit levels.

4. Baseline Analytical Framework

The description of the model will use the standard two-country model regional descriptors, “Home” and “Foreign”. In the case of the simulation, the US will represent “Home” and the trading region will be represented by “Foreign”. The foreign region’s specifications are identical unless otherwise specified and denoted with the use of the “*” notation.

Within each economy the cohort’s age at any random time, t, is given by $t-v$. Agents born at time, v, have a finite lifetime and die at age, D. The cohort variables are denoted by $X(v,t)$, where, v, denotes the cohort vintage and, t, denotes calendar time. The time derivative of a variable at the cohort level is specified as $\partial X(v,t)/\partial t = X_t(v,t)$.

4.1 Demographic Structure

The probability of survival of an individual that is born at time v for age $(t-v)$ is given by a general survival function $S(t-v)=e^{-M(t-v)}$. At the age of birth and death the following survival probabilities are given, $S(0)=e^{-M(0)}=1$ and $S(D)=e^{-M(D)}=0$. The probability of dying at each age, or the instantaneous probability of death, is given by $-S'(t-v)/S(t-v)=\mu(t-v)$.

As stated, the exogenous demographic structure that will be utilized in this paper was developed by Boucekkine et. al. (2002) (BCL) and is given by:

$$e^{-M(t-v)} = \frac{\mu_0 - e^{\mu_1(t-v)}}{\mu_0 - 1} \tag{1.1}$$

The age-dependent instantaneous probability of death increases realistically over an individual’s lifetime and is given by the following:

$$\mu(t-v) = \frac{\mu_1 e^{\mu_1(t-v)}}{\mu_0 - e^{\mu_1(t-v)}} \tag{1.2}$$

Equation (1.1) is a highly tractable and accurate survival function specification. Unlike other tractable survival functions, for instance the perpetual youth specification utilized in Blanchard (1985), this function will produce an age-varying survival probability that realistically increases with age. The BCL demographic structure has two parameters that determine the life expectancy of the agent and the shape of their survival probability distribution. The μ_0 parameter regulates the death rates of young agents. The μ_1 parameter specifies the death rates for the elderly agents. For instance, if the μ_0 parameter increases then the life expectancy of the agents will increase due to a drop in death rates for the youth of the country, while a decrease in μ_1 will cause the death rates for the elderly to drop. Differences in these parameters allows for the modeling of cross-country disparities in overall health, as might be depicted by increased life expectancies. In the numerical simulations below, these parameters will be chosen such that the life expectancy and age-specific survival probabilities will match the data for the included countries.

Figure 2 exhibits the 2010 age survival data and the estimated parameterized BCL survival function for that year. The survival data has been normalized at age 18, corresponding to the birth age of the agents in the simulation. The estimated survival function produces an excellent fit for the majority of the individual's life cycle excluding the longest-living cohorts. Given that these individuals constitute a small percentage of the population, it is unlikely that the BCL approximation will significantly detract from the results of the simulation. As shown by Bruce and Turnovsky (2013b), the BCL function's accuracy is due to the fact that it is an approximation of the highly realistic yet intractable Gompertz function.

4.2 Production

Each country is populated by a representative competitive firm that employs workers and rents capital to produce a homogenous internationally traded good through the following constant returns to scale function:

$$Y(t) = AF\left(K(t), L(t)\right) \qquad (2.1)$$

where $Y(t)$ is the aggregate output at time t, $K(t)$ is aggregate capital located in the Home country, and $L(t)$ is the aggregate supply of labor. The production function satisfies the standard conditions: $F_L > 0, F_K > 0, F_{LK} > 0,$ and $F_{LL}, F_{KK} < 0$. Per-worker output may be expressed by the following:

$$y(t) = \frac{Y(t)}{L(t)} = AF\left(\frac{K(t)}{L(t)}, 1\right) \equiv Af\left(k(t)\right) \qquad (2.2)$$

Output per worker for the home country is denoted by $Af(k)$, where $k(t)$ is the capital-labor ratio and A is the Hicks neutral total factor productivity (TFP) parameter that may differ across countries. The firm rents capital and hires labor such that the following marginal products are equalized with the price of the input:

$$Af'\left(k(t)\right) = r(t) \tag{2.3}$$

$$Af\left(k(t)\right) - Af'\left(k(t)\right)k(t) = w(t) \tag{2.4}$$

Where $r(t)$ is the endogenously determined interest rate and $w(t)$ is the wage rate paid to all workers regardless of age. For tractability I have removed the depreciation rate of capital stock.

$k(t)$ and $k^*(t)$ are the per-worker capital stock located in each country. These capital holdings are owned by both home and foreign agents such that:

$$k(t) = k_d(t) + k_f(t) \tag{2.5}$$

and

$$k^*(t) = k_d^*(t) + k_f^*(t) \tag{2.6}$$

The d or f subscript denotes the domestic or foreign agent, while the “*” notation, denotes where the capital is domiciled.

For the baseline model, it is assumed that agents enter and exit life as workers, causing the labor supply to equate to the population. This translates into the per-worker output equating to the per-capita output and a labor force participation rate equaling one. However, for the augmented model featuring a retirement period, a fraction of the population has exited from the labor force causing the labor supply to be a comprised of the population younger than the mandatory retirement age. Given that $P(t)$ is the size of the population and $L(t)$ is the labor force, the labor force participation rate is given by $l(t) = L(t)/P(t)$. The per-worker variables, $k_d(t)$ and $k_f(t)$, are therefore defined as the aggregate capital owned by the home and foreign agents per home worker.

4.3 The Household

The home country agent maximizes their expected lifetime utility:

$$E\left(U(v)\right) = \int_v^{v+D} \frac{C(v,t)^{1-1/\sigma} - 1}{1-1/\sigma} e^{-\rho(t-v) - M(t-v)} dt \tag{3.1}$$

An individual of cohort v maximizes expected utility generated from the consumption of a traded generic consumption good, $C(v,t)$ and has no bequest motive. σ is the intertemporal elasticity of substitution, ρ is the pure rate of time discount, while $\rho + \mu(t-v)$ is the overall rate of time discount at age $(t-v)$. The consumption choice for the agent has been simplified for two reasons. The first reason is to keep the model transparent and to maximize tractability. The second reason is associated with the terms of trade effect between countries. With the home and foreign countries producing an identical good, the terms of trade remain constant and equal to one. This removes any impact of price adjustments on the dynamics of the current account. This allows for the isolation of the "pure" effect of the demographic and structural transition on the dynamics of the model.

The agent maximizes their utility subject to the instantaneous budget constraint:

$$\frac{\partial K_d(v,t)}{\partial t}+\frac{\partial K_d^*(v,t)}{\partial t}=\left(r(t)+\mu(t-v)\right)K_d(v,t)+\left(r^*(t)+\mu(t-v)\right)K_d^*(v,t)+w(t)-C(v,t) \quad (3.2)$$

Each individual supplies an inelastic labor time quantity set to unity and earns wage, $w(t)$, that is set by the representative firm. The consumer accumulates two types of capital, domestic, $K_d(v,t)$, and foreign, $K_d^*(v,t)$.

I extend the use of a complete and competitive annuities market as set out in Yaari (1965) to the two-country case. The agent receives a domestic premium on the rate of return of their capital equal to their probability of death, $\mu(t-v)$. Each agent in their respective country earns a premium proportional to their accumulated capital level with the agreement that they will transfer all of their wealth to an insurance company at the time of their death. In a two-country framework, there exists a separate annuities market in each country. The agent of one country will hold capital domiciled in both regions, but the insurance company located in the agent's country will distribute the acquired assets upon death only in the country of the individual's origin.

Through maximization of (3.1) subject to (3.2), the following Euler condition will be produced:

$$\frac{C_t(v,t)}{C(v,t)}=\sigma\left(r(t)-\rho\right)\equiv\psi(t) \quad (3.3)$$

$\psi(t)$ denotes the growth rate of consumption. With the assumption of a perfect and competitive annuities market, agents will experience a common consumption growth rate, regardless of age, which adjusts with the endogenously determined international interest rate. If an imperfection is included in the annuities premium, as studied by Heijdra and Mierau (2012) and Bruce and Turnovsky (2013), the consumption growth rate will include the instantaneous probability of death, causing the consumption path to be "hump-shaped". If the instantaneous probability of death is accurate, the consumption path will be concave, reaching a maximum during the middle of an agent's life and declining in the later ages. While many might argue that a concave consumption profile is more realistic, a positive private consumption growth rate has been observed and studied by Tung (2011). My choice of a full annuities market is driven by the tractability gains associated with the ability to remove a bequest structure. The growth rate of consumption is also influenced by the pure rate of time preference, ρ. Adjustments in the rate of time preference will tilt the consumption path over the life cycle. A relatively higher home time preference causes the agent to place a larger value on current consumption. In the international setting, differential time preference rates will influence the NFA position for a country as a preference for higher current or future consumption will effect an agent's age-dependent saving profile.

Using equation (3.3) I derive the path of cohort consumption for the arbitrary time interval from t to τ:

$$C(v,\tau) = C(v,t)e^{\sigma(R(t,\tau)-\rho(\tau-t))} \tag{3.4}$$

Where $R(t,\tau) = \int_t^\tau r(s)ds$.

I define nonhuman wealth of the home agent as their total age-dependent capital holdings consisting of the accumulated capital from both regions:

$$W(v,t) = K_d(v,t) + K_d^*(v,t) \tag{3.5}$$

and for the foreign agent:

$$W^*(v,t) = K_f(v,t) + K_f^*(v,t) \tag{3.6}$$

In order to enforce the solvency of the cohort, the transversality condition is upheld with the equality: $W(v,v+D)=0$. Additionally, because there is no bequest motive in this economy, the agents receive no assets at the time of their birth or $W(v,v)=0$.

Integrating the cohort's instantaneous budget constraint over their finite lifespan, imposing the capital international arbitrage condition, and the transversality condition, I am able to derive the cohort's intertemporal budget constraint:

$$W(v,t) + \int_t^{v+D} w(\tau)e^{-R(t,\tau)-M(\tau-v)+M(t-v)}d\tau = \int_t^{v+D} C(v,\tau)e^{-R(t,\tau)-M(\tau-v)+M(t-v)}d\tau \tag{3.7}$$

Substituting the consumption path (3.4) into the intertemporal budget constraint (3.7), and solving for consumption, I am able to show that cohort consumption at any time t, is a function of the present discounted value of lifetime wages also known as human wealth.

$$C(v,t) = \frac{W(v,t) + \int_t^{v+D} w(\tau)e^{-R(t,\tau)-M(\tau-v)+M(t-v)}d\tau}{\int_t^{v+D} e^{(\sigma-1)R(t,\tau)-\sigma\rho(\tau-t)-M(\tau-v)+M(t-v)}d\tau} \tag{3.8}$$

I define human wealth as:

$$H(v,t) \equiv \int_t^{v+D} w(\tau) e^{-R(t,\tau)-M(\tau-v)+M(t-v)} d\tau \qquad (3.9)$$

The above is the discounted future labor income at time t for cohort of vintage v. Unlike models employing the representative agent framework, the discounting is a function of the uncertain life expectancy of the agent. Human wealth levels across countries are able to diverge due to differences in the mortality of the countries even with symmetric wages.

I define the inverse of the marginal propensity to consume (MPC) as the following:

$$\Delta(v,t) \equiv \int_t^{v+D} e^{(\sigma-1)R(t,\tau)-\sigma\rho(\tau-t)-M(\tau-v)+M(t-v)} d\tau \qquad (3.10)$$

Unlike human wealth, the MPC increases as the life expectancy falls. Differences in the life expectancy across countries will cause asymmetric age-consumption profiles across countries solely through changes in the MPC.

4.4 Aggregation

I define the total population at time t as $P(t)$, the population's crude birth rate as φ, and the population growth rate as n. Abstracting from immigration, the number of living members of cohort v at any time t is given by:

$$P(v,t) = \varphi P(v) e^{-M(t-v)} \qquad (4.1)$$

Where the relative weight of each cohort to the total population is defined as:

$$\frac{P(v,t)}{P(t)} = \varphi e^{-n(t-v)-M(t-v)} \equiv p(t-v) \qquad (4.2)$$

And the dynamics of the weight is given by:

$$\frac{\partial p(t-v)/\partial t}{p(t-v)} = -\left[\varphi + \mu(t-v)\right] \qquad (4.3)$$

The decrease in the relative size of the cohort is due to the overall increase in the population and the death of members of the cohort over time.

In order to obtain the aggregate per-capita variables, I employ the standard aggregator:

$$x(t) = \int_{t-D}^{t} p(t-v) X(v,t) dv \qquad (4.4)$$

To determine the dynamics of the per-capita variable I take the time derivative of (4.4) which yields:

$$\dot{x}(t) = \varphi X(t,t) + \int_{t-D}^{t} p(t-v) X_t(v,t) dv + \int_{t-D}^{t} \frac{\partial p(t-v)}{\partial t} X(v,t) dv \qquad (4.5)$$

and simplifies to:

$$\dot{x}(t) = \varphi X(t,t) + \int_{t-D}^{t} p(t-v) X_t(v,t)dv - \int_{t-D}^{t} \left[n + \mu(t-v) \right] X(v,t)dv \tag{4.6}$$

In (4.5) the following birth and death survival relations have been used: $p(0) = \varphi$ and $p(D) = 0$. Equation (4.6) exhibits the fact that the dynamics are dependent upon the effect of the addition of newborns and the growth of the economy, less the amount given up by the dying.

Using this method I derive the following per-capita home consumption level:

$$c(t) = \int_{t-D}^{t} p(t-v) C(v,t)dv \tag{4.7}$$

Taking the time derivative of (4.7), home consumption dynamics are given by:

$$\dot{c}(t) = \varphi C(t,t) + \left(\sigma \left[r(t) - \rho \right] - n - \mu_c(t - \upsilon_1) \right) c(t) \tag{4.8}$$

Equation (4.9) results from using the mean value theorem of integration on equation (4.7). μ_c is interpreted as the ratio of the consumption given up by the dying to per-capita consumption.

$$\mu_c(t-\upsilon_1) \equiv \frac{\int_{t-D}^{t} \mu(t-v) p(t-v) C(v,t)dv}{\int_{t-D}^{t} p(t-v) C(v,t)dv} \tag{4.9}$$

From equation (4.8) the dynamics of per-capita consumption includes the "generational turnover term", $\Phi(t)$, as defined in Mierau and Turnovsky (2014b):

$$\Phi(t) \equiv \int_{t-D}^{t} \mu(t-v) p(t-v) C(v,t)dv - \varphi C(t,t) + nc(t) \tag{4.10}$$

This term reduces the consumption level due to the arrival of penniless newborns into the economy. It is the difference between the consumption given up by the dying and the relative consumption of newborns to the growth of the per-capita consumption. Additionally, in the small open economy framework, this term would link aggregate consumption to human wealth path, breaking the indeterminacy of the steady state often plaguing models of this type. This is outlined in Ghironi (2006) for a two-country model and in the working paper Oxborrow and Turnovsky (2015) for the small open economy.

Applying the same method, the per-capita home nonhuman wealth dynamics are given by:

$$\dot{W}(t) = \left(r(t) - n \right) W(t) + w(t) - c(t) \tag{4.11}$$

If the available assets were limited to one capital type, (4.11) would reduce to the standard per-capita capital equation of motion of the neoclassical growth model.

Deriving the dynamics of human wealth and the inverse of the MPC at birth is slightly different. Starting out with human wealth, I set $v = t$ and define human wealth at birth as $H(t) \equiv H(t,t)$.

$$H(t) = \int_{t}^{t+D} w(\tau) e^{-R(t,\tau) - M(\tau - t)} d\tau \qquad (4.12)$$

Taking the time derivative and applying the mean value theorem produces the dynamic equation for human wealth:

$$\dot{H}(t) = -w(t) + \left[r(t) + \mu_H(\tau_1 - t) \right] H(t) \qquad (4.13)$$

Where:

$$\mu_H(\tau_1 - t) = \frac{\int_{t-D}^{t} \mu(t-v) p(t-v) H(v,t) dv}{\int_{t-D}^{t} p(t-v) H(v,t) dv} \qquad (4.14)$$

(4.14) is interpreted similarly as (4.9) as the ratio of human wealth given up by the dying to the per-capita human wealth level.

Now looking at the inverse of the MPC out of human wealth, I set $v = t$ and define $\Delta(t) \equiv \Delta(t,t)$.

$$\Delta(t) = \int_{t}^{t+D} e^{(\sigma-1)R(t,\tau) - \sigma\rho(\tau - t) - M(\tau - t)} d\tau \qquad (4.15)$$

Once again using the method outlined by (4.6), I derive the dynamics of the MPC inverse:

$$\dot{\Delta}(t) = -1 + \left[(1-\sigma) r(t) + \sigma\rho + \mu_\Delta(\tau_2 - t) \right] \Delta(t) \qquad (4.16)$$

Where:

$$\mu_\Delta(\tau_2 - t) = \frac{\int_{t-D}^{t} \mu(t-v) p(t-v) \Delta(v,t) dv}{\int_{t-D}^{t} p(t-v) \Delta(v,t) dv} \qquad (4.17)$$

Note that τ_1 and τ_2 are values of τ that will be determined by the mean value theorem.

4.6 Wealth and the Net Foreign Asset Position

I define the NFA position as the difference between the per-capita holdings of foreign capital by domestic agents and the holdings of domestic capital by foreign agents:

$$N(t) \equiv k_d^*(t) - k_f(t) \qquad (5.1)$$

This expression can be related to per-capita wealth by the following:

$$N(t) = W(t) - k(t) \qquad (5.2)$$

The NFA position of either economy can be positive, representing a lender of capital, or negative, representing a borrower of capital and in aggregate must sum to zero.

Taking the time derivative of (5.1) and (2.5):

$$\dot{N}(t) = \dot{k}_d^*(t) - \dot{k}_f(t) \tag{5.3}$$

$$\dot{k}(t) = \dot{k}_d(t) + \dot{k}_f(t) \tag{5.4}$$

Using equations (5.1)-(5.4) I can transform equation (4.11) into the per-capita current account equation:

$$\dot{N}(t) = Af\left(k(t)\right) - nk(t) + \left(r(t) - n\right)N(t) - c(t) - \dot{k}(t) \tag{5.5}$$

Equation (5.5) states that the rate of the accumulation of home NFA's depends on the sum of home production and the per-capita return on those assets less domestic per-capita absorption. Defining the first four terms as national per-capita savings and assuming a constant labor force participation rate, this expression simplifies to the difference of saving and investment.

4.7 The World Equilibrium

The world equilibrium is competitive and involves a fully integrated goods market with no costs of adjustment. Because of the complete integration, I sum equation (4.11) for Home and Foreign and the world per-capita market clearing condition becomes:

$$\dot{k}(t) + \dot{k}^*(t) = Af\left(k(t)\right) + A^* f^*\left(k^*(t)\right) - nW(t) - n^*W^*(t) - c(t) - c^*(t) \tag{6.1}$$

Either region may instantaneously alter their capital stock by entering the world market, however the per-capita wealth of each country remains sluggish. Equation (6.1) states that the sum of accumulated capital, wealth, and private consumption must equal the total production output.

4.8 Steady State Equilibrium

In steady state, the variables are dependent upon age and not calendar time. The dynamics converge to their long run levels. With no additional capital accumulation the per-capita market clearing condition becomes:

$$Af\left(\tilde{k}\right) + A^* f^*\left(\tilde{k}^*\right) = \tilde{c} + \tilde{c}^* + n\tilde{W} + n^*\tilde{W}^* \tag{7.1}$$

With a perfect capital market, capital is allocated until the per-worker marginal products of production are equalized internationally. This causes international interest rates to equalize implying the following firm optimality conditions:

$$\tilde{r} = Af'\left(\tilde{k}\right) = A^* f^{*}'(\tilde{k}^*) \tag{7.2}$$

Differences in wages across countries will be generated through changes in the firm's productivity term and the capital share parameter.

$$\tilde{w} = (1-\alpha)Af\left(\tilde{k}\right) \tag{7.3}$$

Human Wealth and inverse MPC distributions depend upon the age distribution of the population for each country:

$$\tilde{H} = \tilde{w}\int_0^D e^{-\tilde{r}u - M(u)} du \tag{7.4}$$

The inverse MPC at steady state is given by:

$$\tilde{\Delta} = \int_0^D e^{(\sigma-1)\tilde{r}u - \sigma\rho u - M(u)} du \tag{7.5}$$

In order to have a stable population distribution at steady state, I include a condition linking the survival function, death age, D, population growth rate, n, and the birth rate, φ, as specified by Lotka (1998):

$$\varphi\int_0^D e^{-nu - M(u)} du = 1 \tag{7.6}$$

By equating the survival function to zero and solving for the corresponding age I find the age of death. As stated above, an increase in the μ_0 parameter or a decrease in μ_1, will cause the life expectancy for an agent to increase.

$$D = \ln\left(\mu_0\right) / \mu_1 \tag{7.7}$$

Steady state consumption is the per-capita aggregate of the consumption-age profile, where equilibrium consumption over age is increasing by $\sigma(\tilde{r}-\rho)$ from the birth consumption level of $\tilde{C}_0$:

$$\tilde{c} = \tilde{C}_0\tilde{\varphi}\int_0^{\tilde{D}} e^{(\sigma(\tilde{r}-\rho)-n)u - M(u)} du, \quad \tilde{C}_0 = \frac{\tilde{H}}{\tilde{\Delta}} \tag{7.8}$$

The initial consumption level, $\tilde{C}_0$, is derived from the product of the present discounted value of human wealth and the MPC at birth. At birth the individual owns nothing except for the present discounted value of their future wage income stream. Per-capita consumption is constant in steady state, however the individual consumption varies depending on an agent's age. Given the positive consumption growth rate, consumption increases as an individual ages.

Steady state individual per-capita country wealth equations relating consumption, wealth, and output is given by:

$$(\tilde{r}-n)\tilde{W}+\tilde{w}=\tilde{c} \tag{7.9}$$

The above equation represents the steady state current account balance for the home country. Per-capita wealth is composed of the accumulated holdings of home and foreign capital holdings. The no arbitrage condition equates the domestic and foreign interest rate to the international interest rate, however the return to wealth is influenced by the dilution associated with a growing population.

4.9 Linearization

In order to characterize the local dynamics of the economy around the steady state the following equations for home and foreign are linearized for the baseline model (the linearization and the resulting matrices can be found in the appendix for the retirement model): (4.8), (4.11), (4.13), (4.16), and (6.1).

$$\dot{c}(t)=\varphi\frac{H(t)}{\Delta(t)}+\left(\sigma\left[r(t)-\rho\right]-n-\mu_c(t-\upsilon_1)\right)c(t) \tag{8.1}$$

$$\dot{c}^*(t)=\varphi^*\frac{H^*(t)}{\Delta^*(t)}+\left(\sigma^*\left[r(t)-\rho^*\right]-n^*-\mu_c^*(t-\upsilon_1^*)\right)c^*(t) \tag{8.2}$$

$$\dot{H}(t)=-w(t)+\left[r(t)+\mu_H(\tau_1-t)\right]H(t) \tag{8.3}$$

$$\dot{H}^*(t)=-w^*(t)+\left[r(t)+\mu_H^*(\tau_1^*-t)\right]H^*(t) \tag{8.4}$$

$$\dot{\Delta}(t)=-1+\left[(1-\sigma)r(t)+\sigma\rho+\mu_\Delta(\tau_2-t)\right]\Delta(t) \tag{8.5}$$

$$\dot{\Delta}^*(t)=-1+\left[(1-\sigma^*)r(t)+\sigma^*\rho^*+\mu_\Delta^*(\tau_2^*-t)\right]\Delta^*(t) \tag{8.6}$$

$$\dot{W}(t)=\left(r(t)-n\right)W(t)+w(t)-c(t) \tag{8.7}$$

$$\dot{W}^*(t)=\left(r(t)-n^*\right)W^*(t)+w^*(t)-c^*(t) \tag{8.8}$$

$$\dot{k}(t)+\dot{k}^*(t)=Af\left(k(t)\right)+A^*f^*\left(k^*(t)\right)-nW(t)-n^*W^*(t)-c(t)-c^*(t) \tag{8.9}$$

The dynamics associated with equations (8.1) to (8.9) are functions of the mortality variables μ_c, μ_H, and μ_Δ for both countries. These variables adjust slightly as the demographic age distributions change over the transition. The steady state relationships of these variables for the US are given by:

$$\tilde{\mu}_c=\frac{\tilde{\varphi}\tilde{H}}{\tilde{\Delta}\tilde{c}}+\sigma\left(\tilde{r}-\rho\right)-n \tag{8.10}$$

$$\tilde{\mu}_H = \frac{\tilde{w}}{\tilde{H}} - \tilde{r} \tag{8.11}$$

$$\tilde{\mu}_\Delta = \frac{1}{\tilde{\Delta}} - (1-\sigma)\tilde{r} - \sigma\rho \tag{8.12}$$

I approximate these mortality variables as constants during the transition as their influence would be minimal.[5] The mortality values for the baseline model's initial equilibrium for the US are $\mu_c = 0.0168$, $\mu_H = 0.0012$, and $\mu_\Delta = 0.002$. Their values for the final equilibrium are given by $\mu_c = 0.0166$, $\mu_H = 0.0008$, and $\mu_\Delta = 0.0015$. During the full demographic transition of the baseline model, the average change in the American mortality variables is only 0.0004, justifying their approximation as constants during the transition.

Noting the equalization of the marginal products of capital, I transform the above system of dynamic equations into functions of per-capita domestic capital, $k(t)$. This allows for the simplification of the system of equations, as shown in the Appendix. The resulting linearized matrix is found below in vector form, a more detailed version is found in the appendix.

$$\begin{pmatrix} \dot{c}(t) \\ \dot{c}^*(t) \\ \dot{H}(t) \\ \dot{H}^*(t) \\ \dot{\Delta}(t) \\ \dot{\Delta}^*(t) \\ \dot{W}(t) \\ \dot{W}^*(t) \\ \dot{k}(t) \end{pmatrix} = \Omega \begin{pmatrix} c(t) - \tilde{c} \\ c^*(t) - \tilde{c}^* \\ H(t) - \tilde{H} \\ H^*(t) - \tilde{H}^* \\ \Delta(t) - \tilde{\Delta} \\ \Delta^*(t) - \tilde{\Delta}^* \\ W(t) - \tilde{W} \\ W^*(t) - \tilde{W}^* \\ k(t) - \tilde{k} \end{pmatrix} \tag{8.13}$$

This system will have a unique bounded path if the linearized matrix of dimension 9x9, given by Ω, yields a positive determinant with two negative and seven positive eigenvalues. The stable eigenvalues are represented by λ_1 and λ_2. I assume that the per-capita nonhuman wealth for each country, $W(t)$ and $W^*(t)$, are both sluggish. Capital stock, consumption, human wealth at birth, and the MPC at birth all respond instantaneously to shocks.

4.10 General solution

From the linearized matrix above, (13.14), I derive the following solution:

[5] For a more detailed analysis of the impact of varying mortality variables, see Mierau and Turnovsky (2104b).

$$W(t) = \tilde{W} + C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \tag{9.1}$$

$$W^*(t) = \tilde{W}^* + C_1 V_{1W^*} e^{\lambda_1 t} + C_2 V_{2W^*} e^{\lambda_2 t} \tag{9.2}$$

$$c(t) = \tilde{c} + C_1 V_{1c} e^{\lambda_1 t} + C_2 V_{2c} e^{\lambda_2 t} \tag{9.3}$$

$$c^*(t) = \tilde{c}^* + C_1 V_{1c^*} e^{\lambda_1 t} + C_2 V_{2c^*} e^{\lambda_2 t} \tag{9.4}$$

$$H(t) = \tilde{H} + C_1 V_{1H} e^{\lambda_1 t} + C_2 V_{2H} e^{\lambda_2 t} \tag{9.5}$$

$$H^*(t) = \tilde{H}^* + C_1 V_{1H^*} e^{\lambda_1 t} + C_2 V_{2H^*} e^{\lambda_2 t} \tag{9.6}$$

$$\Delta(t) = \tilde{\Delta} + C_1 V_{1\Delta} e^{\lambda_1 t} + C_2 V_{2\Delta} e^{\lambda_2 t} \tag{9.7}$$

$$\Delta^*(t) = \tilde{\Delta}^* + C_1 V_{1\Delta^*} e^{\lambda_1 t} + C_2 V_{2\Delta^*} e^{\lambda_2 t} \tag{9.8}$$

$$k(t) = \tilde{k} + C_1 V_{1k} e^{\lambda_1 t} + C_2 V_{2k} e^{\lambda_2 t} \tag{9.9}$$

Where the arbitrary constants, C_1 and C_2, are derived by solving the system:

$$\begin{pmatrix} 1 & 1 \\ V_{1W^*} & V_{2W^*} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} W_0 - \tilde{W} \\ W_0^* - \tilde{W}^* \end{pmatrix} \tag{9.10}$$

The variable, V_{ij}, is the eigenvector coefficient associated with the stable eigenvalue $i = 1$ or 2 for λ_1 or λ_2 respectively and the variable $j = W(t), W^*(t), c(t), c^*(t)$ etc. In solving for the arbitrary coefficients I assume that the system starts from an initial nonhuman wealth position, W_0 or W_0^*, for each country. Each country's initial wealth position may be different depending on the similarities of cross-country demographic characteristics.

4.11 Net Foreign Asset Position Dynamics

Starting from the dynamic equation (5.5) describing the home country's current account as a function of the accumulation of capital with the assumption of a constant labor force participation rate:

$$\dot{N}(t) = f(k) - nk + \big(r(t) - n\big) N(t) - c(t) - \dot{k}(t) \tag{9.11}$$

Substituting in the dynamic world equilibrium equation, I simplify the expression to a simple separable first order differential equation:

$$\begin{aligned} \dot{N}(t) &= (1-\chi) f(k) - \chi f^*(k^*) - nk + \big(r(t) - n\big) N - c(t) \\ &+ \chi n W(t) + \chi n^* W^*(t) + \chi c(t) + \chi c^*(t) \end{aligned} \tag{9.12}$$

Where χ is defined as:

$$\chi \equiv \left[\left(\frac{\alpha A}{\alpha^* A^*} \right)^{\frac{1}{\alpha^*-1}} \left(\frac{\alpha - 1}{\alpha^* - 1} \right) k^{\frac{\alpha - \alpha^*}{\alpha^* - 1}} + 1 \right]^{-1} \tag{9.13}$$

Linearizing this around the steady state:

$$
\begin{aligned}
\dot{N}(t) = & \left[(1-\chi) f'(\tilde{k}) + f''(\tilde{k})\tilde{N} - n - \chi \frac{\alpha-1}{\alpha^*-1}\frac{\tilde{k}^*}{\tilde{k}}\tilde{r}\right](k(t)-\tilde{k}) \\
& + \chi n\left(W(t)-\tilde{W}\right) + \chi n^*\left(W^*(t)-\tilde{W}^*\right) \\
& + (\chi-1)(c(t)-\tilde{c}) + \chi\left(c^*(t)-\tilde{c}^*\right) \\
& + \left(f'(\tilde{k})-n\right)\left(N(t)-\tilde{N}\right)
\end{aligned} \tag{9.14}
$$

Defining:

$$\phi_k \equiv (1-\chi) f'(\tilde{k}) + f''(\tilde{k})\tilde{N} - n - \chi \frac{\alpha-1}{\alpha^*-1}\frac{\tilde{k}^*}{\tilde{k}}\tilde{r} \tag{9.15}$$

$$\epsilon_1 \equiv \left(\phi_k V_{1k} + \chi n + \chi n^* V_{1W^*} + (\chi-1)V_{1c} + \chi V_{1c^*}\right)C_1 \tag{9.16}$$

$$\epsilon_2 \equiv \left(\phi_k V_{2k} + \chi n + \chi n^* V_{2W^*} + (\chi-1)V_{2c} + \chi V_{2c^*}\right)C_2 \tag{9.17}$$

Using the transversality condition I solve for the NFA position of the home country for any arbitrary time period t.

$$N(t) = \tilde{N} + \frac{\epsilon_1}{\lambda_1 - (\tilde{r}-n)}e^{\lambda_1 t} + \frac{\epsilon_2}{\lambda_2 - (\tilde{r}-n)}e^{\lambda_2 t} \tag{9.18}$$

The NFA position for either country can be either positive or negative representing a net lender or debtor, respectively. In the situation where the two countries are identical with respect to structural parameters and demographic characteristics the NFA position for each country will be zero. Only when there is an asymmetry in the saving behavior will the NFA position for the country become nonzero. International asset market clearance requires that the summation of the international NFA position be zero, or $N(t) + N^*(t) = 0, \forall\, t$. With the NFA's path defined, I can derive the current account transition path, which is defined as the time derivative of the NFA position.

Taking the time derivative of (9.18) I derive the following current account time path for the home country for any arbitrary time t:

$$CA(t) \equiv \dot{N}(t) = \frac{\lambda_1\epsilon_1}{\lambda_1 - (\tilde{r}-n)}e^{\lambda_1 t} + \frac{\lambda_2\epsilon_2}{\lambda_2 - (\tilde{r}-n)}e^{\lambda_2 t} \tag{9.19}$$

4.12 Gradual versus Instantaneous Transition

Demographic change does not occur rapidly. In order to compare the gradual versus instantaneous adjustment of key parameters, the baseline transitional paths are augmented to include the ongoing change of population growth and the birth rate over a period of time. Additionally, in order to contrast an immediate versus ongoing change in productivity, TFP adjustments have also been gradually modeled. This modeling technique forces the variable to follow a continuous transition, eliminating discrete jumps. Because of the interconnected nature of the two countries, a change in either of these rates for one country will have an impact on both countries.

Within the baseline dynamics, I have built in the dynamic process accounting for the gradual population growth, birth rate, and productivity change with the following functions:

$$n(t) = \tilde{n} + \left(n_0 - \tilde{n}\right)e^{-\theta t} \tag{10.1}$$

$$n^*(t) = \tilde{n}^* + \left(n_0^* - \tilde{n}^*\right)e^{-\theta^* t} \tag{10.2}$$

$$\varphi(t) = \tilde{\varphi} + \left(\varphi_0 - \tilde{\varphi}\right)e^{-\omega t} \tag{10.3}$$

$$\varphi^*(t) = \tilde{\varphi}^* + \left(\varphi_0^* - \tilde{\varphi}^*\right)e^{-\omega^* t} \tag{10.4}$$

$$A(t) = \tilde{A} + \left(A_0 - A\right)e^{-at} \tag{10.5}$$

$$A^*(t) = \tilde{A}^* + \left(A_0^* - A^*\right)e^{-a^* t} \tag{10.6}$$

Where θ and ω are the rate of change of the population growth and birth rate per year for the home country and a is the rate of change for the TFP parameter, A. The impact of a gradual increase in population growth versus an immediate change (corresponding to an infinite θ, ω, or a value) on the dynamics of the countries will be substantial.

5. Augmented Model Including Retirement

This section details the required modifications to the model in order to include a social security system and retirement period. As stated, with a retirement period incorporated into the individual's life cycle, the population no longer equals the labor supply. I therefore conduct the equilibrium analysis in per-worker rather than per-capita terms. A shown below, this will impact the dynamics through adjustments in the labor force participation rate. An identical preference structure is used in this analysis, but the agent's budget constraint will now differ. In order to incorporate the national pension system, an additional governmental sector must be included in the model. Lastly, the dynamics of human wealth will need to be approximated. This is due to the discontinuity in the human wealth equation caused by the retirement period.

5.1 The Household

While preferences remain identical to the baseline model, the instantaneous budget constraint takes on a slightly different form due to the presence of a retirement period and the social security benefit payment:

$$\frac{\partial K_d(v,t)}{\partial t}+\frac{\partial K_d^*(v,t)}{\partial t}=\left(r(t)+\mu(t-v)\right)K_d(v,t)+\left(r^*(t)+\mu(t-v)\right)K_d^*(v,t) \\ +(1-\tau_s)w(t)I(R)+B(t)\left(1-I(R)\right)-C(v,t) \quad (11.1)$$

$I(R)$ is an indicator function equaling one during the employment period. Similar to the baseline setup, during the employment period, the agent earns the aggregate wage regardless of their age. The individual is required to pay an income tax, τ_s, in order to fund the national pension system. Given that labor is supplied inelastically, the tax rate's distortionary impact is minimized. The individual earns this after-tax wage until the date of their mandatory and exogenous retirement age, R. Following this period, the agent receives income from the social security benefit, $B(t)$. The agent receives this benefit for a period of expected length $D-R$.

Through the implementation of the transversality condition and the inclusion of (3.4) the individual consumption at an arbitrary time t may once again be written in terms of capital wealth, human wealth, and the MPC:

$$C(v,t)=\frac{W(v,t)+H(v,t)}{\Delta(v,t)} \quad (11.2)$$

The human wealth equation is now augmented to account for the retirement period and social security system.

$$H(v,t)\equiv\int_t^{v+R}(1-\tau_s)w(\tau)e^{-R(t,\tau)-M(\tau-v)+M(t-v)}d\tau+\int_{v+R}^{v+D}\mathrm{B}(\tau)e^{-R(t,\tau)-M(\tau-v)+M(t-v)}d\tau \quad (11.3)$$

Unlike the baseline human wealth definition, (11.3) includes the discounted after tax aggregate wage, $(1-\tau_s)w(t)$, and the pension benefit, $\mathrm{B}(t)$, that the individual earns during the associated period of employment or retirement.

5.2 The Government

A government sector is included to conduct the transfer of income from the employed to the retirees. The government is represented by a typical pay-as-you-go transfer scheme. To fund this system the government taxes labor income at rate τ_s for all working individuals. All individuals are taxed at the same rate while they are employed regardless of age. Taxes are used solely to fund the benefit that retired agents receive and therefore the government maintains a balanced budget each period given by the following relation:

$$\int_{t-R}^{t} p(t-v)\tau_s w(t)dv = \int_{t-D}^{t-R} p(t-v)B(t)dv \tag{11.4}$$

The benefit retirees receive, $B(t)$, is proportional to the aggregate wage rate such that, $B(t) = \beta w(t)$. In the two-country setting each country can differ with respect to their retirement age, tax rate, and retirement benefit. Given that the government is maintaining a balanced budget, if the simulated tax rate is matched to the observed value, the benefit will be endogenously determined.

5.3 Equilibrium Modifications

Due to the fact that the population diverges from the labor supply, differentiation is needed between the per-capita and per-worker aggregate. For any general cohort variable, $X(v,t)$, I can derive its per-worker variable, $\bar{x}(t)$:

$$\bar{x}(t) = \frac{1}{l}\int_{t-D}^{t} p(t-v)X(v,t)dv \tag{11.5}$$

Where l is the labor force participation rate and is defined as:

$$l = \int_{t-R}^{t} p(t-v)dv = \frac{\varphi\int_0^R e^{-nu-M(u)}du}{\varphi\int_0^D e^{-nu-M(u)}du} \tag{11.6}$$

This specifies the fraction of the employed population and exhibits the fact that, due to the demographic steady state, the labor force participation rate is time-invariant as long as the exogenous retirement age and demographic structure is constant throughout the transition. The labor force participation rate is dependent upon the retirement age, population growth rate, and the survival function. A longer period of retirement will decrease the participation rate while an increase in the population growth rate will increase the rate due to additional agents entering the economy as workers. A change in the mortality of the population could have an ambiguous effect on the participation rate. While in one case, a decrease in mortality can lengthen the period of employment due to improved health; it also may increase the length of retirement. In this framework, the retirement age is set exogenously in order to match observed country characteristics. Therefore the length of the working period will be fixed exogenously, causing a decrease in mortality to increase the retirement length and decrease the labor force participation rate.

Equilibrium human wealth in the retirement model is given by:

$$\tilde{H} = \int_0^R (1-\tau_s)\tilde{w}e^{-\tilde{r}(u)-M(u)}du + \int_R^D \beta\tilde{w}e^{-\tilde{r}(u)-M(u)}du \tag{11.7}$$

If the benefit proportion of the wage, β, is equal to the proportion of after-tax wages kept by the individual, $1-\tau_s$, then the augmented human wealth relation will simplify to an after tax version of the baseline human wealth definition.

5.4 Modifications to the Dynamics

The time derivative of (11.5) yields the per-worker dynamics of the model:

$$\dot{\bar{x}}(t) = \frac{\varphi}{l}X(t,t) + \frac{1}{l}\int_{t-D}^{t} p(t-v)X_t(v,t)dv + \frac{1}{l}\int_{t-D}^{t}\frac{\partial p(t-v)}{\partial t}X(v,t)dv - \frac{\dot{l}(R)}{l(R)}\bar{x}(t) \tag{11.8}$$

The final term accounts for the adjustment of the labor supply to a change in the retirement age or the demographic structure. This term is naturally absent in the baseline model due to the fact that the labor force participation rate is constant. The final term will be nonzero if the retirement age, R, or the demographic variables vary during the transition. Assuming an instantaneous adjustment for the demographic parameters and the exogenous retirement age, l will remain fixed at its final value and the final term above will drop out.

The time derivative of the human wealth equation (11.3) is given by:

$$\begin{aligned}\dot{H}(t) = (1-\tau_s)\Big[& w(t+R)e^{-R(t,t+R)-M(R)} - w(t) + \int_t^{t+R}[r(t)+\mu(\tau-t)]w(\tau)e^{-R(t,\tau)-M(\tau-t)}d\tau\Big] \\ +\beta\Big[& -w(t+R)e^{-R(t,t+R)-M(R)} + \int_{t+R}^{t+D}[r(t)+\mu(\tau-t)]w(\tau)e^{-R(t,\tau)-M(\tau-t)}d\tau\Big]\end{aligned} \tag{11.9}$$

Due to the discontinuity associated with equation (11.9), we use the following approximated version of the dynamics of human wealth represented by:

$$\dot{H}(t) = -(1-\tau_s)w(t) + [r(t)+\mu_H(\tau-t)]H(t) - [\beta-(1-\tau_s)]w(t)e^{\left[\frac{\dot{w}(t)}{w(t)}-r(t)\right]R} \tag{11.10}$$

As stated in Mierau and Turnovsky (2015), the final term in (11.10) is an approximation of the wage income evolution occurring at the anticipated retirement age. A detailed derivation can be found in the Appendix.

6. Baseline Simulation

To determine the impact of international asymmetric demographic trends I include numerical simulations modeling the transitional dynamics around the steady state. In order to simulate the interaction between these six countries in a two-country framework, I develop a population-weighted average of the trading partners for multiple exogenous factors including: the survival function parameters, the annual population growth rate, the total factor productivity, and the production capital share parameter. I parameterize the BCL survival function to match the included countries for the years 1980 and 2010.[6] The BCL survival function parameters, μ_0 and μ_1, have been estimated with nonlinear least squares using age-survival data from the Human Mortality Database.[7] I focus the analysis on the impact of cross-country differences in demographic characteristics including differences in birth rates, population growth rates, and the decline in mortality. Additionally, I estimate the impact of the demographic and structural transition on the natural rate of wealth inequality present in an OLG economy through the calculation of the wealth Gini coefficient. Lastly, I analyze the impact of the presence of a retirement period and social security structure on the NFA position of the countries.

Table 2 summarizes the key baseline parameter values including the demographic parameter estimates. Given that my focus is the impact of demographic change, I set the parameter values such that I isolate its effect. To that end, the intertemporal elasticity of substitution is set to 0.5, consistent with the range given by Guvenen (2006). The pure rate of time preference is 0.035. Output is produced by a Cobb-Douglas function, $F(K,L) = AK^{\alpha}\bar{L}^{1-\alpha}$, for each country, where $\bar{L}$ denotes the inelastic labor supply, A is the productivity parameter that has been initially normalized for both countries to 1.0, and the capital share parameter is initially given by $\alpha = 0.35$. Rate variables, impacting the transitional speed for the gradual transition paths in the baseline model, have all been equalized at 5.0 percent.

The estimated BCL function tracks observed mortality data remarkably well for most Western nations as it does here. The function produces an accurate approximation of the survival data resulting in an adjusted R^2 value of 0.99. As a childhood period is not included in

[6] Arguments can be made for the insertion of the observed mortality data directly. This would require the analysis to be performed in discrete time. While this is an option, given the close fit of the BCL function with the data, the parameterization of the survival function does not significantly harm the accuracy of the analysis. Additionally, with this continuous time framework, the analysis can be executed with the use of differential equation rather than difference equation techniques. Lastly, I am performing a two-country analysis that is similar to a comparative-static approach. The use of this modeling framework allows the analysis to be performed in a much more direct and simple manner.

[7] Human Mortality Database. University of California, Berkeley (USA), and Max Planck Institute for Demographic Research, Rostock (Germany). Available at www.mortality.org.

the model, the estimation is performed for the age interval from 18 to 90. In order to parameterize the survival function for the region, I estimate the mortality parameters for each trading partner. I then create an overall population weighted average utilizing weights associated with the population of each country for the specified age interval. It is important to note that the decrease in the population growth rate during this time period is a trend exhibited by most countries excluding France, whose growth rate remained at 0.5 percent, and the United Kingdom, which experienced an increase in their growth rate from 0.3 to 0.8 percent. Since this model lacks immigration, the matched population growth rates are reflected in the birth rate. The initial population growth rate for both countries is set to 1.0 percent.

6.1 Decrease in the Mortality Rate

I first focus on the impact of a decrease in mortality (an increase in the survival probability for each age) on the NFA position of the countries. Figure 3 exhibits the transitional dynamics associated with the asymmetric decrease in mortality between the US and the trading region. Over the 30-year period, the life expectancy for the US increases from 74.1 to 78.6 years and from 76.4 to 81.5 years for the region due to the decline in mortality. The lifetime extension generates a significant negative NFA position for the US. It is important to note that, while the overall population growth rate is constant at 1.0 percent, due to the change in the mortality parameters, the fertility rate drops for the US from 1.93 to 1.85 percent and from 1.89 to 1.81 percent for the trading region. This has a slightly positive effect on the per-capita variables.

The longer life interval generates an increase in wealth due to the extended lifespan during which capital accumulates. Given that each country experiences an increase in expected life, per-capita wealth for all countries increase over time. The immediate impact of the life expectancy increase is to decrease consumption in order to satisfy the goods market clearing constraint for the longer-lived population. As agents accumulate capital, output increases, and consumption will follow suit. The difference in saving across countries manifests itself in the NFA position. Due to their relatively shorter life span, the US enjoys slightly higher per-capita consumption and a lower savings level, which translates into a negative and decreasing American NFA position. The decline in mortality causes the NFA position to decrease by 13 percent during the 30-year interval. While this change is substantially lower than the amount observed by the US during that period, as shown in Table 5, the model generates a negative and declining position solely because of the demographic transition, something that has not been accomplished in other studies, see Backus et al. (2014). In the long run, measured from the initial to the final steady states, the NFA position for the US declines by 44 percent.

6.2 Full Demographic Adjustment

In Figure 4 I include the adjustment of the overall population growth rates along with the mortality decline as seen in Figure 3. During the demographic transition, the population growth rate for the US falls from 1.0 to 0.8 percent. The weighted population growth rate for the region falls from 0.52 to 0.26 percent. Due to the fact that the gradual dynamics adjust to account for the change in the population growth and birth rates, differences in the instantaneous and gradual transition paths emerge to a greater degree. For the simulation I set the adjustment speed to a reasonable 5.0 percent.

Even with the addition of the differential transition of the population growth rates, the instantaneous dynamic paths remain remarkably similar to those found in Figure 3, however the magnitude of the NFA change falls slightly. Within the first 30 years the American NFA position declines by 7.0 percent, while the long run decrease measures only 26 percent. This is due to the fact that, as the population growth rate falls, fewer newborns enter the economy causing an increase in the proportion of wealthier older agents who save a greater level of wealth each period. This effect is magnified by an increase in the per-capita return on wealth as shown by equation (8.7). The increase in the return is the result of the decrease in the per-capita dilution of assets. These combined influences place a positive pressure on the NFA position for the US, limiting the fall in the NFA position. Nevertheless, the lower overall life expectancy and higher mortality relative to the trading region preserves the negative American NFA position.

Taken in isolation, an asymmetric decline in the population growth rate between two initially symmetric countries will cause the wealth levels of the affected country to increase in the long run. The decline in the growth rate will cause the country's per-capita consumption to experience an initial jump and a long run increase. If the transitional growth rate is restricted to adjust gradually, as described in Section 4.12, the gradual consumption path initially overshoots the instantaneous path, before experiencing a positive but declining transition necessary to clear the goods market. This overshooting is caused by the expectation that capital will be accumulated in the long run, but population growth rates are still constrained in the present. As the population growth rate adjusts to its long run value, wealth will begin to accumulate once more and consumption will transition to its final higher steady state value. The foreign region, reacting to the cross-country asymmetry in population growth rates, experiences the opposite effect. Wealth levels ultimately fall and consumption will initially decrease and reach a new lower steady state.

6.3 Structural Change

Figures 5-7 illustrate the impact of structural changes in addition to the full demographic effects (a decline in mortality and the population growth rates). The two structural changes being analyzed are an asymmetric increase of the total factor productivity, shown in Figure 5, and an increase in the capital share parameter of the production function, shown in Figure 6. For the trading region I use a population-weighted average for the TFP and production share parameters. The combined effect of both a TFP and share parameter change is found in Figure 6. Over the 30-year time period, the US experienced a 34 percent increase in productivity, while the trading region experienced an average productivity increase of 4.25 percent. The trading region's increase in productivity is largely due to the substantial increase experienced by the UK of 31 percent and the modest 8 percent gain experienced by Japan. All other countries included in the region experienced a productivity decrease. These values can be seen in Table 8. Additionally, following the global trend, the labor share of production has fallen for both the US and the trading region by 8.1 percent and by 6.9 percent, respectively.

The permanent TFP gain immediately increases consumption for the US, however the most substantial change is the jump and slope reversal of the NFA transitional path. Taken by itself, a permanent change in productivity will not generate a NFA position as it will not impact the saving behavior of a country. A rise in productivity increases the accumulation of capital in the affected country while simultaneously increasing consumption, wages, and wealth by the same percentage. As shown by Ghironi et al. (2005), a productivity change will only impact the NFA holdings if there previously exists a current non-zero asset position. For their study, they use heterogeneity in time preferences to induce a nonzero NFA position in order to analyze the effect of a productivity shock. For this study, an asymmetric productivity shock amplifies the nonzero NFA position generated by the asymmetric demographic transition.

Including the productivity changes with differential demographics causes the NFA path to remain negative similar to the path exhibited by the data. However, the TFP change generates an overshooting of the initial jump that causes the path to exhibit a positive trend, opposite of what has been observed for the US during that period. The substantial negative jump in the NFA position is due to the large increase in the investment into the US. With the added impact of the region's relatively high saving rate, associated with the longer life expectancy and low population growth rate, the long run NFA position for the US stays negative and decreases by 61 percent.

The capital shares of production for the US and the region are similar in magnitude, 0.432 and 0.439 for the US and the trading region in 1980 respectively, and 0.467 and 0.47 in 2010.[8] During the 30-year time frame, the shares increased as production further emphasized capital inputs, with the trading region weighting capital slightly more during the entire period. The isolated impact of an increase in the home country's capital share is to draw in investment from overseas, decreasing the NFA position of the country. Ceteris paribus, the observed share positions cause the US to generate a positive NFA position. However the convergence of the shares cause a decrease in the American NFA position by 53 percent. If the transition of the capital shares is included with the full demographic change, in the first 30 years the NFA position for the US falls by 85 percent as calculated from the initial equilibrium. The American NFA position experiences a decrease by 124 percent in the long run.

Lastly, Figure 7 shows the impact of the combined structural effects. The capital share adjustment exacerbates the overshooting, leading to a transitional path that experiences a more negative position, but a steeper positive slope. In the long run, the NFA position for the US falls by 202 percent. The increase in productivity benefits the countries as they both experience an increase in consumption and an accumulation of wealth.

6.4 Asymmetric Time Preference

In the previous analysis the pure rate of time preference was held to be constant throughout the transition. However, relative differences between preference rates will generate a nonzero NFA position due to the international disparity in the valuation of current versus future consumption.[9] In their empirical cross-country study, Wang et al. (2011) find that rates of time preference vary substantially across countries. They test this by surveying economics students from 45 countries, questioning whether the student would prefer to receive a smaller payment immediately or a larger payment in the future. With the exception of France, which was not included in their sample, a higher proportion of students from all of the countries included in the trading region (Canada, France, Germany, Japan, and the UK) would prefer to wait than would students from the US. In order to see the effect of differential time preferences in the face of

[8] Capital shares are taken from Nieman (2013).

[9] See Buiter (1981) and Ghironi et al. (2005) among others.

demographic change, I adjust the time preference rate for the US to 4.0 percent (an increase from 3.5 percent) while holding the region's rate constant at 3.5 percent. Ceteris paribus, I find that a relative increase in the rate of time preference generates a significant negative NFA position. If the American rate of time preference is held constant at 4 percent, the long run decrease in the NFA position amounts to a fall of 16.29 percent in response to the full demographic transition. If the American rate of time preference is set to decrease from 3.5 to 4.0 percent over the 30-year period, the long run NFA position falls by 271 percent. The substantial change in the American time preference towards current consumption augments the impact of the relatively higher saving behavior of the region, additionally decreasing the NFA position for the US.

6.5 Wealth Inequality across Countries

I now turn to estimating the inequality of the age-dependent distribution of assets in each country and the effect of the demographic transition and structural change on that allocation. Assuming international arbitrage with the rate of return on capital and using the definition of nonhuman wealth, the individual's budget constraint at the steady state is given by the following:

$$\dot{\tilde{W}}(u) = \left(\tilde{r} + \mu(u)\right)\tilde{W}(u) + \tilde{w} - \tilde{C}(u) \tag{12.1}$$

Integrating this and using (7.8), I solve for the steady state wealth distribution across the cohorts:

$$\tilde{W}(u) = e^{\tilde{r}u + M(u)}\left\{\tilde{w}\int_0^u e^{-\tilde{r}u - M(u)}du - \frac{\tilde{H}}{\tilde{\Delta}}\int_0^u e^{((\sigma-1)\tilde{r}-\sigma\rho)u - M(u)}du\right\} \tag{12.2}$$

From the steady state individual asset equation, (12.2), I am able to derive the cohort wealth Gini coefficient for each country; a measure of inequality attributed to individuals being at different points of their savings life cycle, Mierau and Turnovsky (2013). This measure relates how unequal the distribution of assets is across the generations. According to Davies et al. (2007), the concentration of wealth within countries is often high. A common interval of wealth Gini coefficients for a country will lie between 0.6-0.8. In comparison, the Gini coefficient associated with disposable income often falls within the 0.3 to 0.5 interval. Using the estimated demographic characteristics and the observed structural changes from 1980 to 2010, the simulated model produces the following "natural" wealth Gini coefficients for the US and the region as shown by Table 3.

Obviously the observed Gini coefficients, as shown in Table 7, are far different from the smaller estimated statistics for the modeled countries. This is to be expected due to the fact that this inequality estimate is solely measuring the wealth inequality associated with an age-dependent saving difference across individuals. I find that the mortality decline, irrespective of the observed productivity change, causes a slight decrease in inequality. The extended lifetime accompanying a decrease in mortality increases the proportion of wealthy individuals in the economy. Additionally, because I hold the population growth rate constant while decreasing mortality, there is a minor fall in the birth rate necessary to maintain the steady state demographic distribution. This slight fall in the birth rate enhances the downward pressure on the Gini coefficient. Conversely, a pure decrease in old age mortality will increase the Gini coefficient. Including the fall in the population growth rate substantially decreases the younger cohort sizes relative to the older ones, thereby "flattening" out the relative cohort distribution, naturally decreasing inequality. This causes there to be a significantly fewer number of poor, younger individuals in the economy relative to wealthy, older individuals significantly decreasing the Gini estimate.

Moving to the included Gini statistics associated with the structural changes, I first consider the productivity shock. As stated, each country experiences a TFP increase during the time period, with the US experiencing a relatively larger gain. This increase in productivity triggers an accumulation of capital that generates a complementary effect on wages in both regions. This wage-driven income effect on inequality is negligible due to the fact that all individuals receive the same wage. However, the increase in the return of capital benefits the older capital-laden individuals, which causes an increase in inequality. Overall, the fall in the population growth rate, as stated above, dominates this effect and the Gini coefficient falls slightly. The reduction in inequality associated with the change in the capital shares is also impacted by the greater magnitude of the fall in population growth rates. Removing the population growth rate change causes inequality to slightly increase similarly to the TFP shock. Finally, the inclusion of all the demographic and structural changes causes inequality to decline, with the overall demographic change overcoming the effect of the change in the productivity and capital shares. In summary, due to the similarity across individuals, structural changes have a small effect on the level of wealth inequality. However, demographic changes, primarily those impacting the number of newborns entering the economy, have a sizeable effect.

7. Retirement Model Simulation

I augment the previous analysis by including a model incorporating a retirement period and social security system. This framework allows for the determination of the effect of an asymmetric aging of the workforce of the economies, as exhibited by the falling labor force participation rate. Additionally, with the inclusion of a pay-as-you-go social security structure, I am able to examine how the decline in retirement benefits and increase in income taxes will impact the saving behavior over the life cycle. The maximum retirement period for each economy is 25 years, assuming the agent reaches their maximum age of death, with an expected retirement period of 10 years. The modeled ages of retirement and pension replacement rates are exogenous and have been set to match the observed values. With the exclusion of Canada, over the 30-year time period all countries experienced a fall in their pension benefit level, with the US experiencing a decline of 12 percent and the trading region experiencing a decline of 29.7 percent.[10]

The gradual dynamic analysis has been removed for this model for simplification. The dynamics of the macroeconomic equilibrium can be approximated by the system of per-worker equations given by (4.8), (4.11), (4.16), (5.5), and (11.10) for the home country and a similar system of equations for the foreign region. As shown above, the dynamic equations associated with per-worker consumption and wealth both include the labor supply adjustment term. With an instantaneous adjustment, the labor force participation rate is constant during transition and the term is removed.

7.1 Mortality Decrease – Retirement Model

Figure 8 displays the transitional dynamics associated with a decrease in mortality. In anticipation of the extended retirement period during which no wage income is earned, per-capita savings for both countries increase. The slightly longer life expectancy for the region causes its agents to save relatively more than Americans. Due to this, the per-capita wealth for the region increases by 7.3 percent, while the US only experiences an increase of 6.3 percent. In consequence of the increase in the saving rate, agents decrease their consumption expenditure and per-capita consumption for both countries falls. The relative frugality of the region causes the NFA position for the Americans to develop a negative and decreasing value over time. Initially, the American NFA position decreases by 11.45 percent. In the short run, from the

[10] Pension rates can be found in Table 8.

initial equilibrium to year 30, the per-capita NFA position for the US decreases by 23.15 percent. In the long run, Americans experience an NFA position decrease of 39.44 percent.

7.2 Demographic Transition – Retirement Model

The reduction of the population growth rate naturally causes the labor force participation rate to decline significantly due to the diminished number of workers entering the economy. As a result of the fall in the population growth rates, the labor participation rate declines by 5.9 and 7.62 percent for the US and the region, respectively. Due to the trading region's relatively lower population growth rate during the entire period, their labor supply remained well below that of the US. The region's low population growth rate and relatively greater decline decreases the per-capita dilution of wealth. In response to the total demographic adjustment, the per-capita NFA position experiences a relatively larger initial jump relative to the previous isolated mortality shock. As a result of the full transition, the American's NFA position experiences an initial fall by 18.4 percent, a 30-year decrease by 24.82 percent, and a long run decrease by 34 percent. Taken in isolation, a decrease in the relative population growth rate increases the level of the per-capita variables. This will naturally increase per-capita wealth levels, driving savings overseas and generating a positive NFA position.

7.3 Demographic Transition with Social Security Structure

Figure 10 displays the per-capita dynamics associated with the full demographic change and the introduction of a social security structure. Including the social security system entails a couple of important adjustments to the model. The most visible change is the establishment of a governmental sector collecting a wage income tax and transferring the proceeds to the retired. In this model, the tax rate is found endogenously through the interaction of the per-capita wage rate, identical for all workers within a country, and the labor force participation rate, which determines the tax base. While the exogenously set replacement rates experience a fall over time, due to the increasing proportion of retirees and the falling population growth rates, income taxes must increase in order to compensate. In light of this, the tax rate for the US experiences a long run increase by 12.7 percent. Due to the dramatic fall in benefits over the period of 22.9 percent, the region's tax rate modestly increases by 3.3 percent.

In this simulation, I match the observed effective retirement ages for the included countries. During the 30-year period, each country experiences a decline in retirement age by approximately 1.0 year. Taken in isolation, the country that enacts an increase in the retirement age will experience a long run increase in per-capita wealth and consumption. Due to the longer time frame during which an individual can accumulate savings and the shorter expected retirement period, the saving rate falls for the country, causing the country to generate a negative NFA position.

As shown by Figure 10, the inclusion of the social security system causes the US to experience a positive but declining NFA position for the following reasons. The relatively low and declining population growth rate of the trading region decreases the number of workers entering into the economy. This increases the per-capita wealth, which pushes the region to become an international lender. Furthermore, the relatively higher life expectancy increases the savings of the region above that of the Americans, placing further positive pressure on the NFA position. The fall in the effective retirement age decreases the time interval during which the region's inhabitants can accumulate savings and increases the expected length of the retirement period. This increases their relative saving rate and drives the American NFA position slightly more negative. Finally, the substantial decline in the relative pension benefit reverses the trend and pushes the American NFA position to a positive but declining level. The massive reduction in benefits for the trading region's retirees causes the region's agents to save and the American NFA position to return to a negative position once again. Taken together, these effects cause the long run American NFA position to fall by 318 percent.

8. Conclusion

This paper has developed a two-country neoclassical growth model including a realistic demographic structure for the main purpose of analyzing the impact of asymmetries of demographic characteristics on cross-country interdependence. Unlike many models in international economics, transitional dynamics are generated primarily by the evolution of demographic variables. In this analysis I have exhibited a few major points. The first is that a relative increase in the life expectancy of the population will exert a positive force on the NFA position of a country. This is generated by the fact that individuals who experience an extension of their lifetime will increase their savings. In the international setting, the higher relative saving rate of the country will naturally lead it to be a lender in the international capital market. This result holds whether or not a retirement period is included in the model. The retirement period will limit the interval during which agents are able to accumulate wealth, but an increase in the saving behavior of the workers will cause the overall result to hold.

Secondly, the population growth rate of a country will exert significant pressures on the per-capita dynamics of the NFA position. A relative increase in the population growth rate will generate a decline in the NFA position because of the increase in the dilution of per-capita assets. An increase in the proportion of poor youths will push the per-capita wealth levels down, thereby exerting a pressure on the country to borrow internationally.

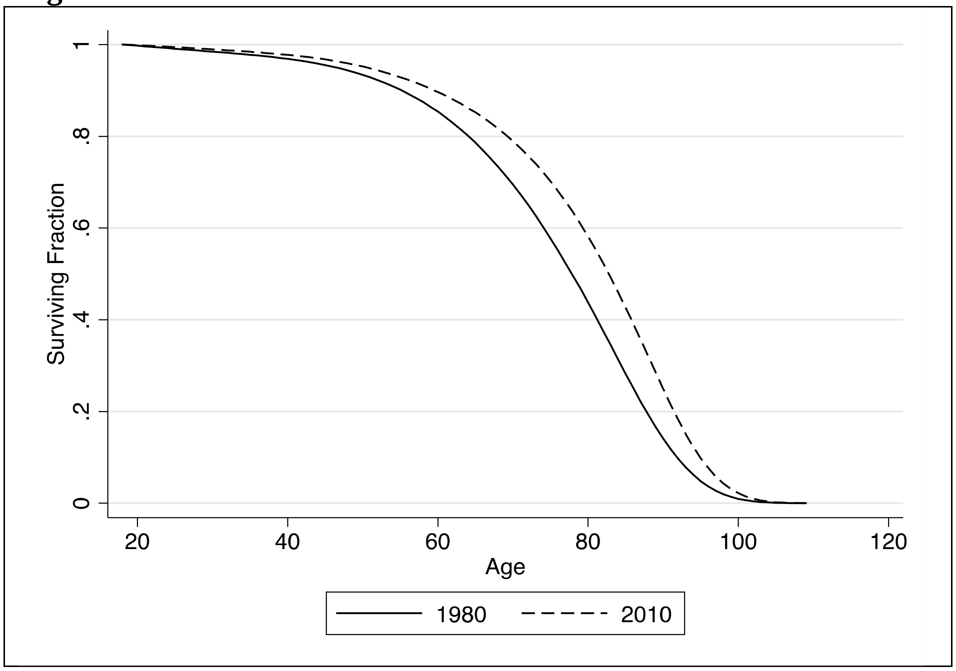
The third is that the rate of time preference has a significant impact on the NFA position. A relatively high time preference rate generates a negative NFA position due to the population's partiality towards current consumption. With the empirical results from the study by Wang et al. (2011) suggesting that the US might experience a relatively higher time preference rate than the other countries included in this study, I model the impact of a higher American preference rate during the 30 year period. The transition to a higher preference rate for the US causes a significant decline in the long run American NFA position, reinforcing the effect of the region's high saving rate.

These findings relate to the natural wealth inequality for the baseline model as follows. Inequality declines with a fall in the population growth rate due to the smaller poor youth population. A fall in mortality causes an increase in inequality due to the presence of longer living wealthy individuals; however, with a constant population growth rate, the birth rate will fall to maintain the demographic steady state thereby decreasing inequality.

While the demographic framework of this model is complex, I realize that the overall model does simplify many structures to a sizeable extent. One important extension of this work is to compare the resulting equilibria and dynamics with a limited and asymmetric annuities structure present in each country. Not only would this potentially increase the accuracy of consumption across generations, as it would cause the consumption-age profile to be "hump shaped", but the implications for the consumption and saving behavior and therefore the NFA position would be significantly influenced. Additionally, the inclusion of a labor choice and age-dependent productivity would also strongly affect the nature of the cross-country interdependence. International differences in the age-productivity profile would reinforce demographic asymmetries across countries. The relative proportion of youth in the population would inherently change the productivity of the country's factors of production impacting international capital flows. With the inclusion of elastic labor, due to the complementarity of labor and capital in the production process, cross-country capital flows would influence the labor-leisure choice of the worker, directly impacting the aggregated savings of the economy.

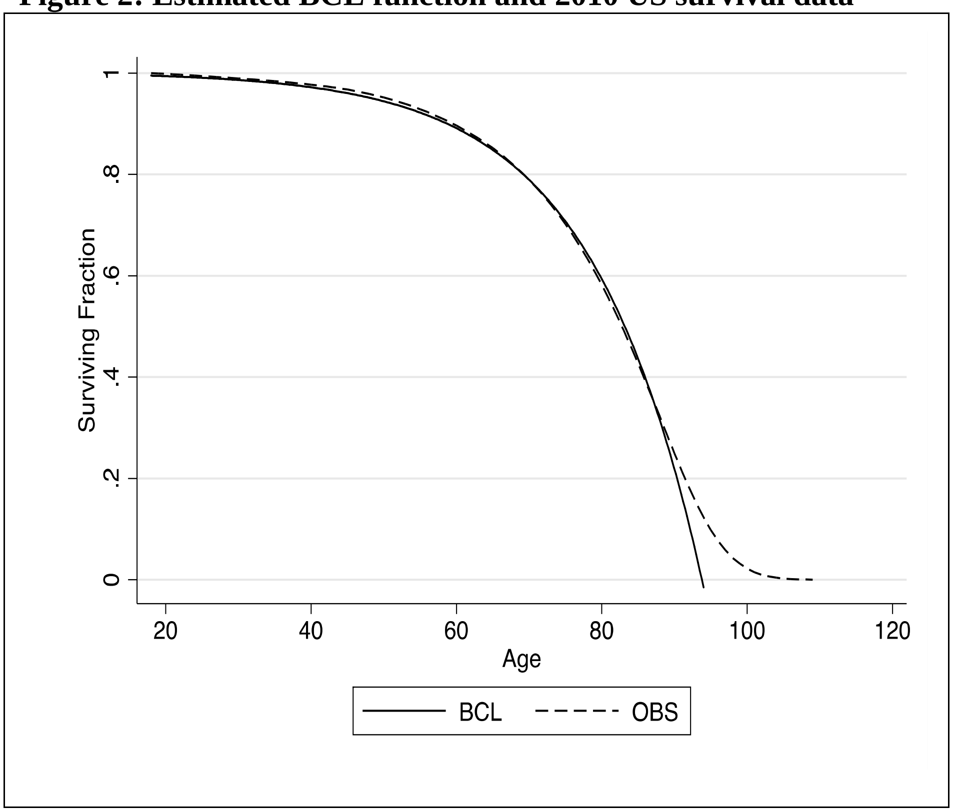
These complexities would increase the realism of the analytical framework and add additional international asymmetries to analyze.

Figure 1: US observed survival data 1980-2010



Data retrieved from www.mortality.org

Figure 2: Estimated BCL function and 2010 US survival data



Data retrieved form: www.mortality.org

Figure 3: Baseline Model - US (Home) and multi-country (Foreign) mortality decrease

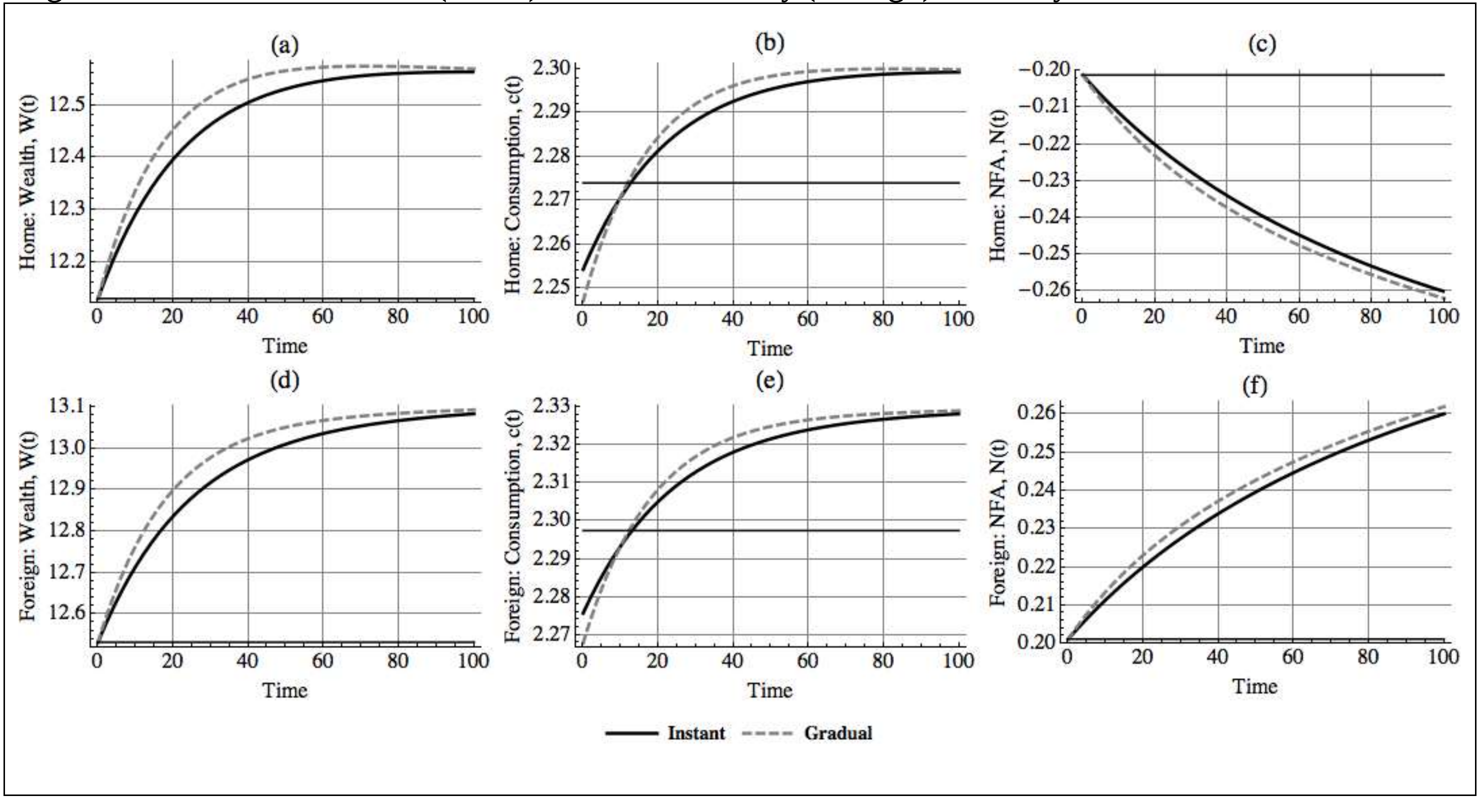


Figure 4: Baseline Model - US (Home) and multi-country (Foreign) mortality decrease population rate decline

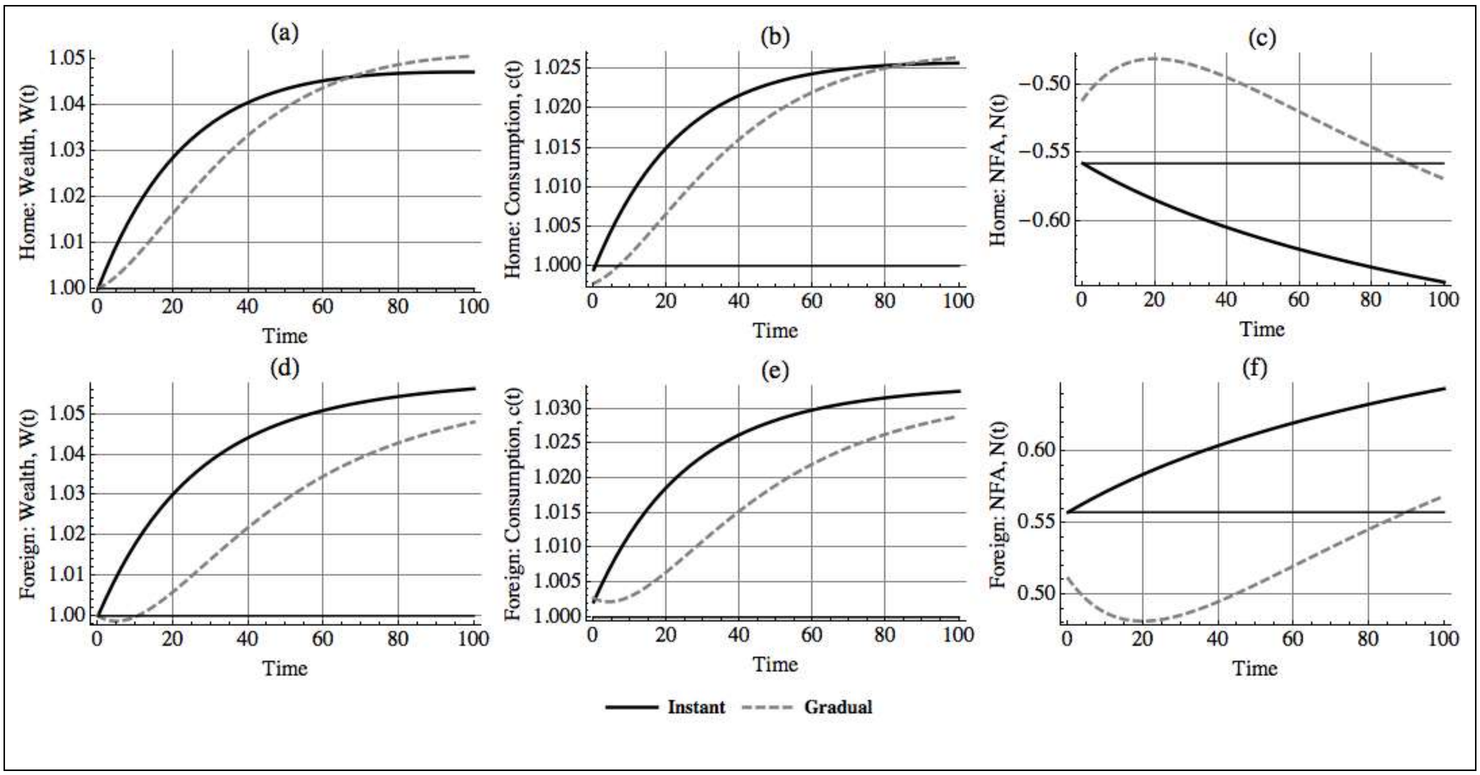


Figure 5: Baseline Model - US (Home) and multi-country (Foreign) productivity increase

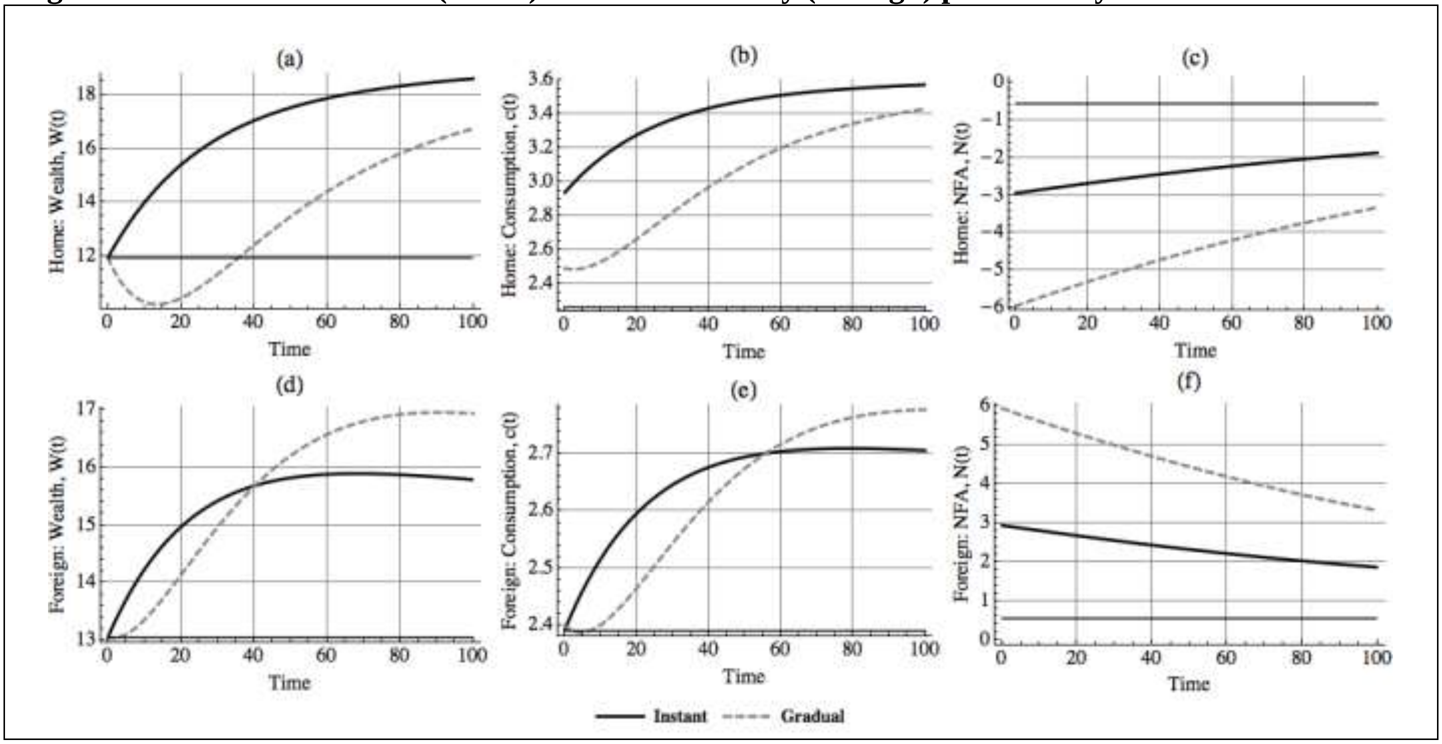


Figure 6: Baseline Model - US (Home) and multi-country (Foreign) capital share increase

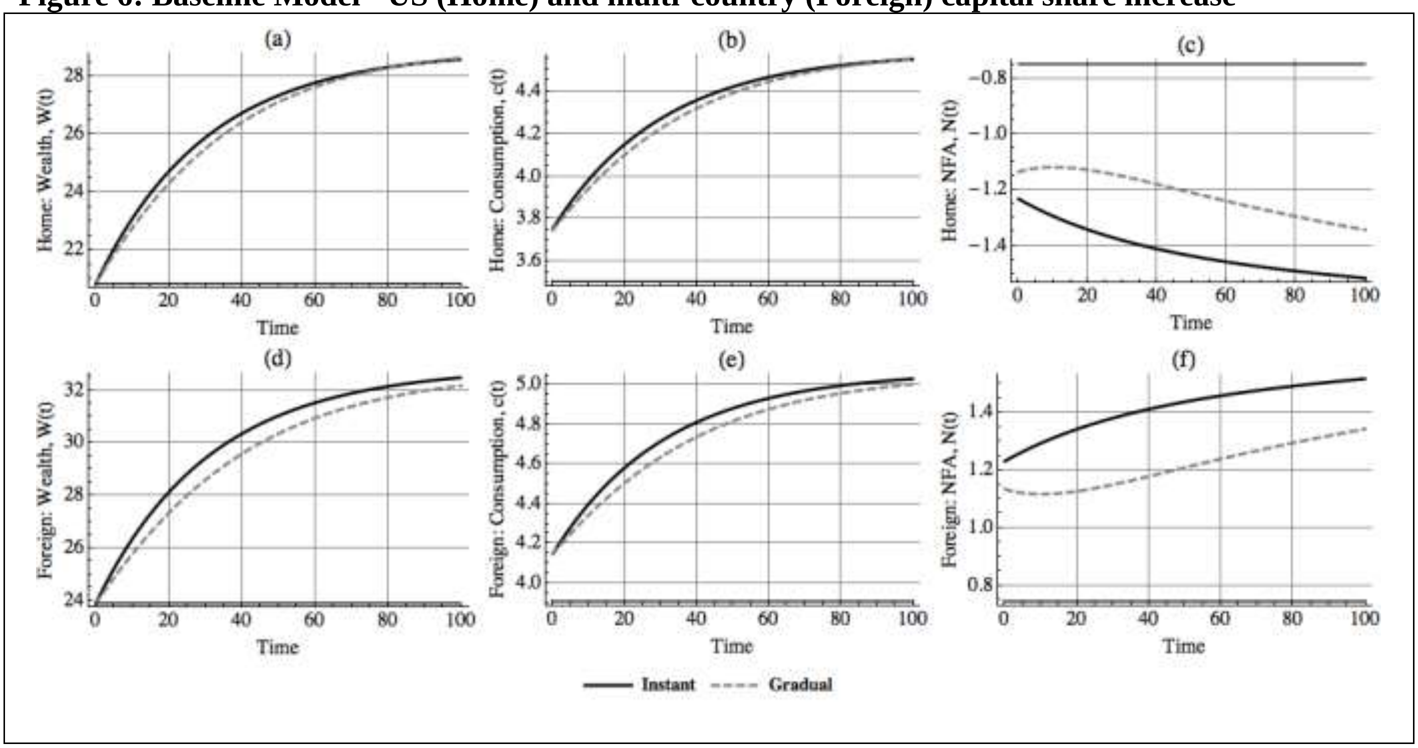
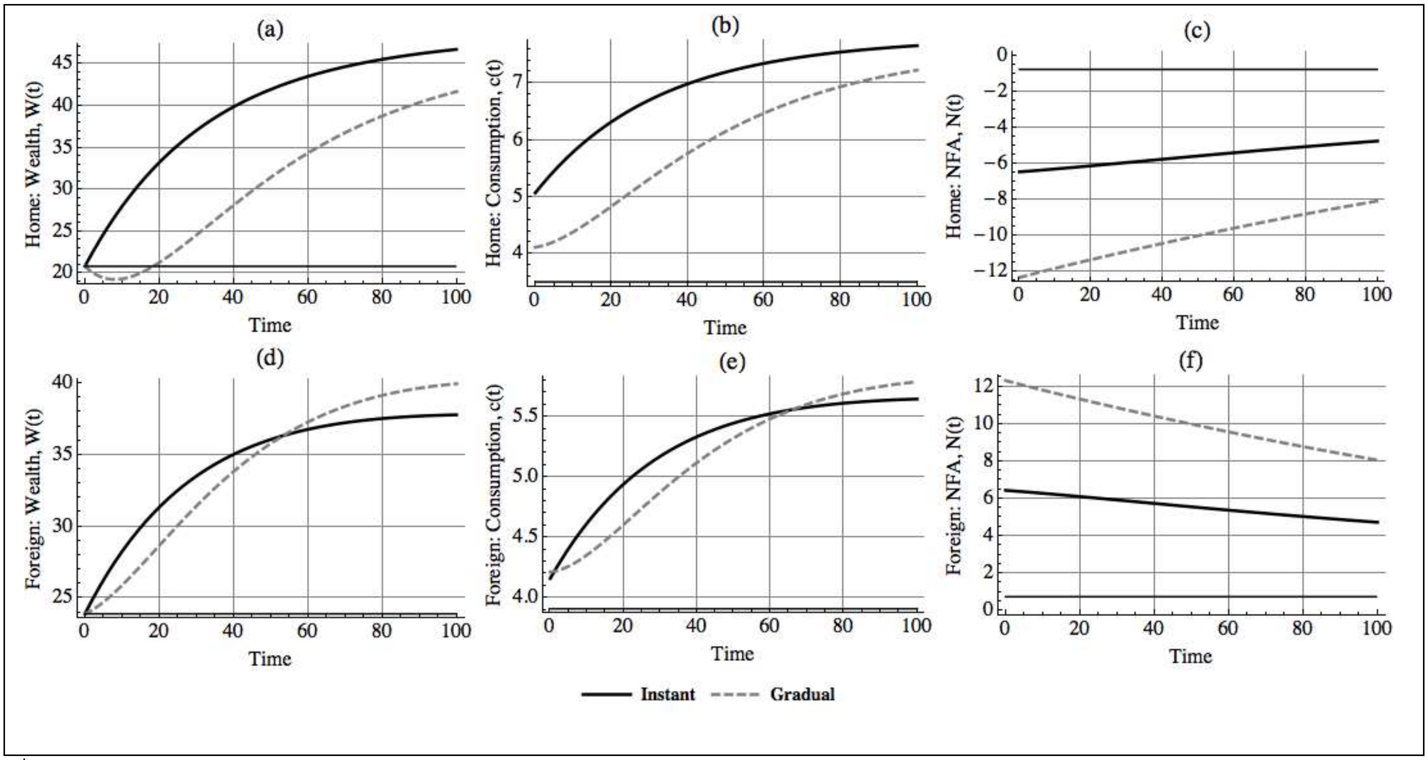


Figure 7: Baseline Model - US (Home) and multi-country (Foreign) productivity and capital share increase



*TFP data from PWT release 8.1.

**Capital share data available at: http://faculty.chicagobooth.edu/loukas.karabarbounis/research/index.html.

Figure 8: Retirement model - mortality decrease

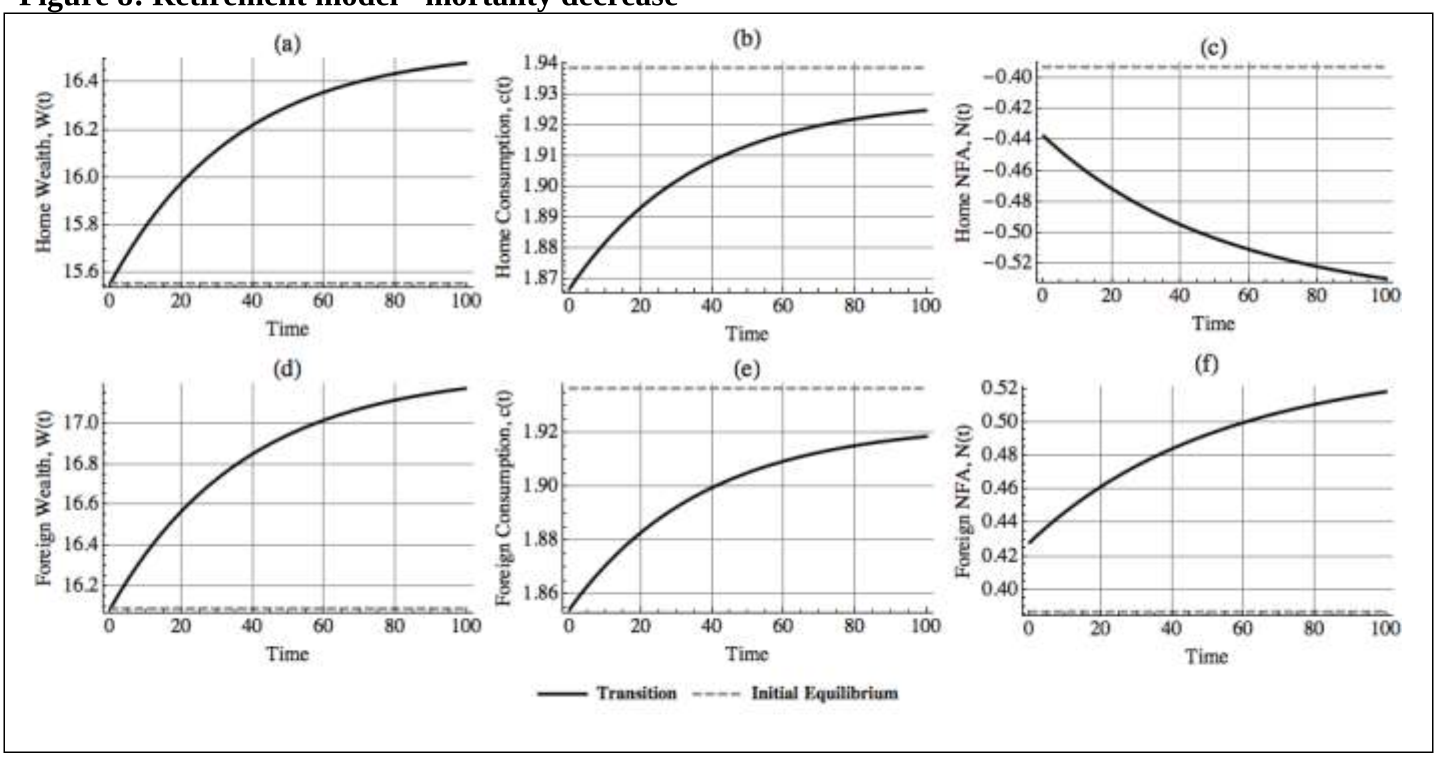


Figure 9: Retirement model - mortality and population growth rate decrease

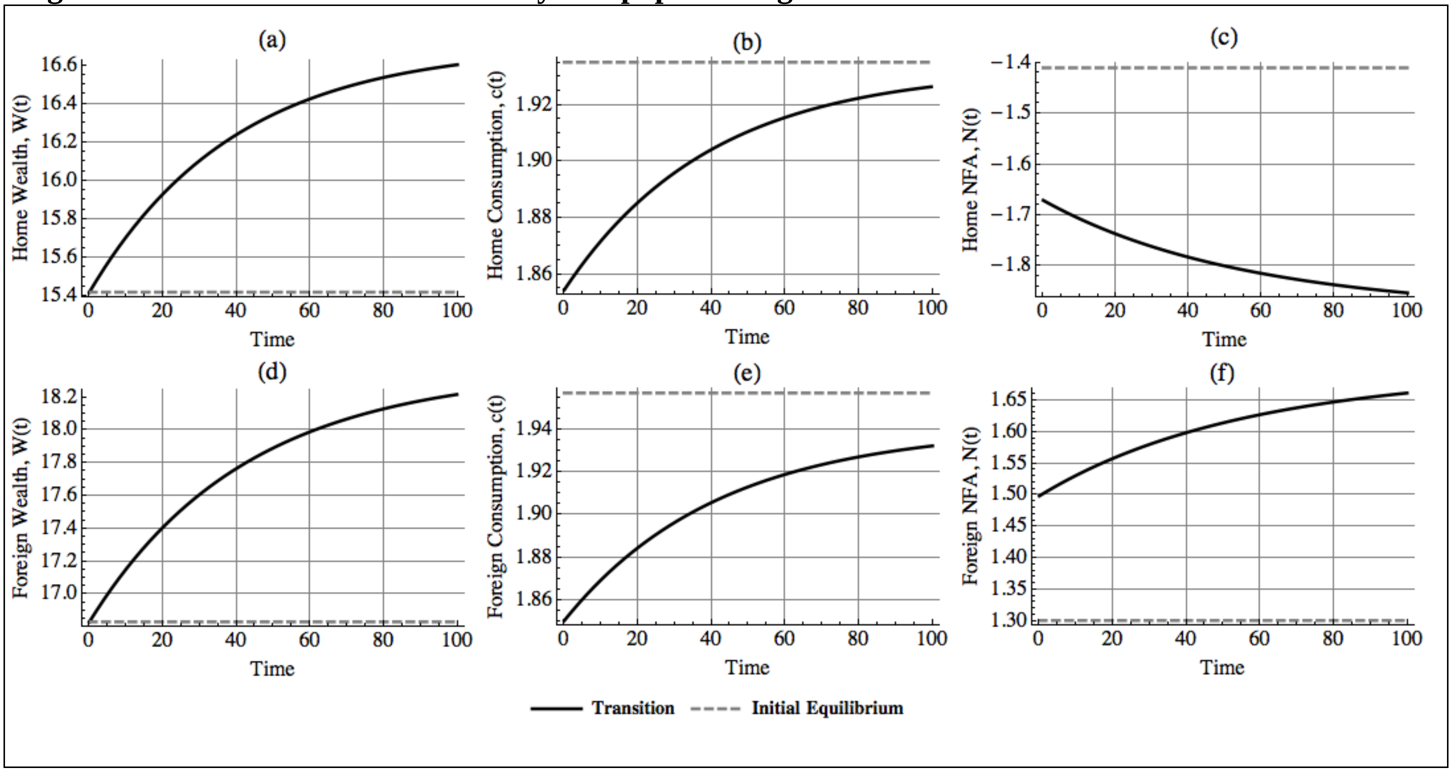


Figure 10: Retirement model - Demographic change with social security structure

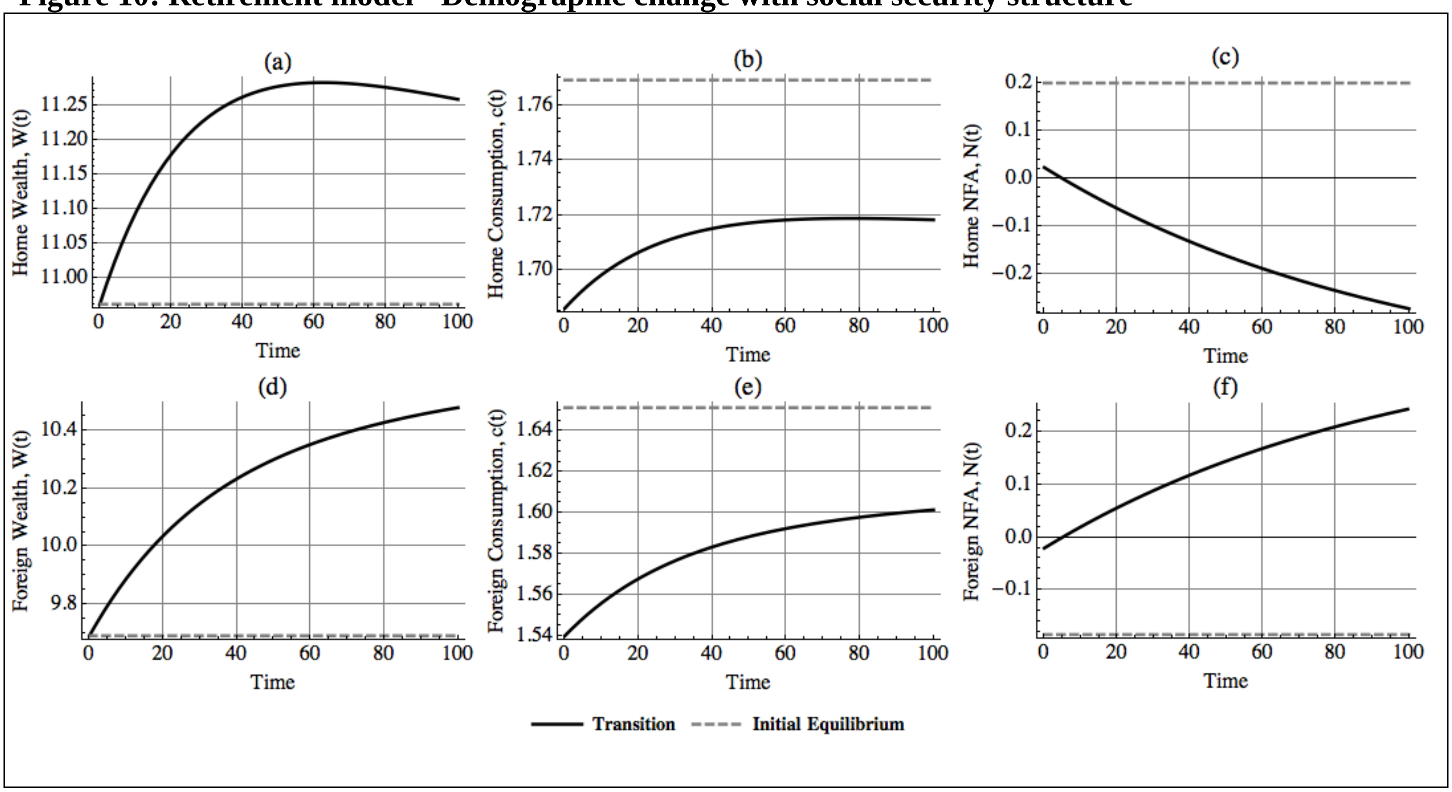
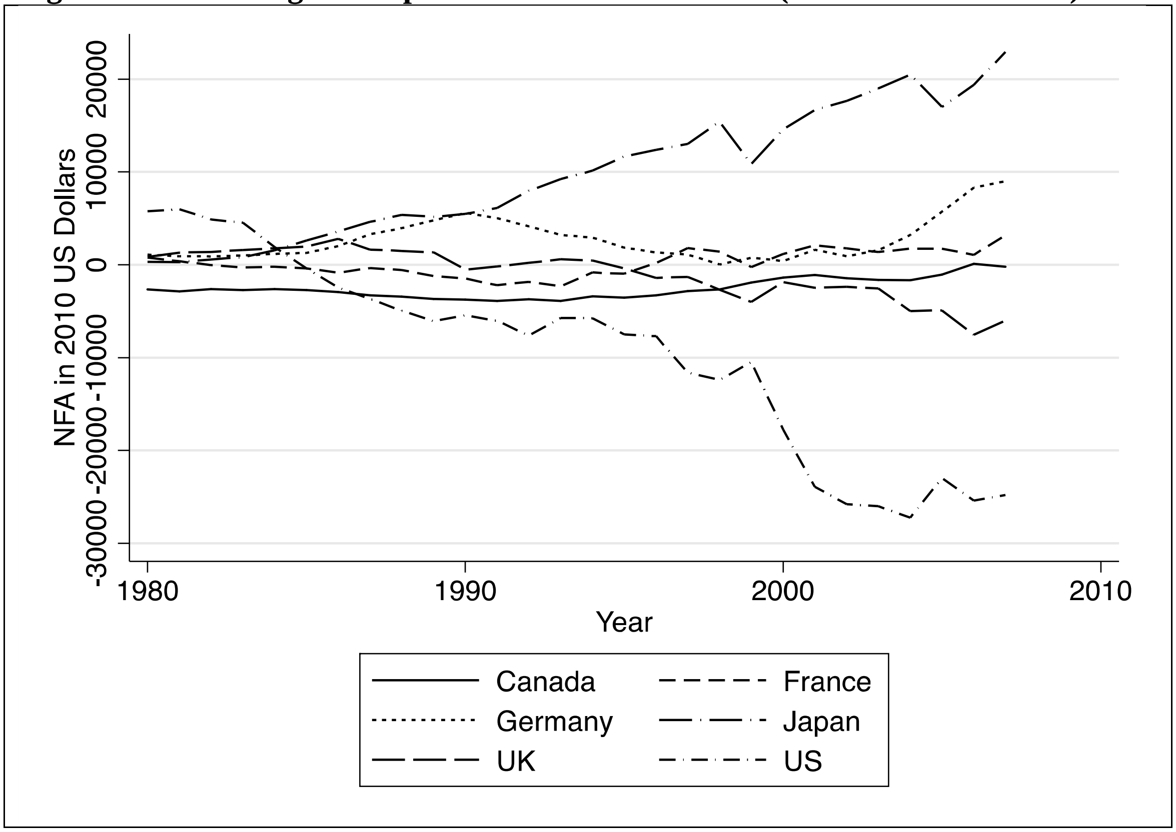


Figure 11: Net foreign asset position in 2010 US dollars (hundreds of millions)



Data from Lane, Milessi, and Ferreti (2007).

Table 1: Cross-country demographic characteristics

	Birth Life Expectancy		Pop. Growth Rate		Fertility Rate	
Country	1980	2010	1980	2010	1980	2010
Canada	75.1	80.9	1.3	1.1	1.7	1.6
France	74.1	81.7	0.4	0.5	1.9	2.0
Germany	72.7	80.0	0.2	-0.2	1.4	1.4
Japan	76.1	82.8	0.8	0.0	1.8	1.4
UK	73.7	80.4	0.1	0.8	1.9	1.9
US	73.7	78.5	1.0	0.8	1.8	1.9

Data retrieved from the World Bank World Development Indicators.

Table 2: Parameter and demographic values

Preference Parameters:	Value:	
Time preference rate, ρ	3.5 %	
Intertemporal Elasticity of Substitution, σ	.5	
Initial Specification (1980 Estimate):	USA	Region
Life expectancy at age 18, L_{18}	74.1	76.4
Implied maximum age, D	91.1	91.6
Youth mortality, μ_0	184.1073	378.0784771
Old age mortality, μ_1	0.0572393	0.064820507
Birth Rate (Implied), φ	1.93 %	1.6 %
Population growth rate, n	1 %	0.518 %
TFP, A	1	1
Capital share production parameter, α	0.432	0.439
Retirement age, R	66.4	65.9
Pension replacement rate, β	0.44	0.515
Final Specification (2010 Estimate):	USA	Region
Life expectancy at age 18, L_{18}	78.6	81.5
Implied maximum age, D	93.8	94.6
Youth mortality, μ_0	451.192	1323.2226
Old age mortality, μ_1	0.0651832	0.075948
Birth Rate (Implied), φ	1.73 %	1.37 %
Population growth rate, n	0.8 %	0.267 %
TFP, A	1.34	1.0425
Capital share production parameter, α	0.467	0.469
Retirement age, R	65.5	64.9
Pension replacement rate, β	0.387	0.397
Speed Parameters:	USA	Region
Population growth rate speed, θ	5 %	5 %
Birth rate speed, ω	5 %	5 %
TFP rate speed, a	5 %	5 %

Table 3: Gini coefficient adjustment

	US		Region	
	Initial	Final	Initial	Final
Mortality Decline	0.3563	0.354	0.3552	0.3544
Mort. & Pop. Rate	0.3547	0.3412	0.3304	0.3146
Demographic* & TFP	0.3547	0.3417	0.3304	0.3151
Demographic & Shares	0.3769	0.3706	0.3518	0.3424
Demographic, TFP, & Shares	0.3769	0.3715	0.3518	0.3432

* Demographic includes the mortality decline and population growth rate fall.

Table 4: Baseline Model - Equilibrium values for each shock

		Output	Output*	Total Capital	Total Capital*	Cons.	Cons.*	Wealth	Wealth*	NFA	NFA*	r
Mortality	Initial	2.409	2.409	12.331	12.331	2.274	2.298	12.13	12.532	-0.201	0.201	0.068
	Final	2.443	2.443	12.831	12.831	2.298	2.331	12.54	13.123	-0.291	0.291	0.067
Mortality & n	Initial	2.42	2.42	12.497	12.497	2.263	2.391	11.939	13.054	-0.558	0.558	0.068
	Final	2.465	2.465	13.16	13.16	2.319	2.474	12.455	13.864	-0.704	0.704	0.066
Dem. & A	Initial	2.42	2.42	12.497	12.497	2.263	2.391	11.939	13.054	-0.558	0.558	0.068
	Final	3.86	2.623	20.551	13.966	3.644	2.643	19.65	14.867	-0.901	0.901	0.066
Dem. & α	Initial	3.77	3.976	21.59	23.153	3.506	3.909	20.845	23.898	-0.745	0.745	0.075
	Final	4.929	5.041	30.442	31.314	4.573	5.08	28.775	32.981	-1.667	1.667	0.076
Dem., A & α	Initial	3.77	3.976	21.59	23.153	3.506	3.909	20.845	23.898	-0.745	0.745	0.075
	Final	8.508	5.435	52.351	33.635	7.937	5.51	50.104	35.882	-2.247	2.247	0.076
Dem. & ρ	Initial	2.420	2.420	12.497	12.497	2.263	2.391	11.939	13.054	-0.558	0.558	0.068
	Final	2.427	2.427	12.597	12.597	2.203	2.528	10.527	14.667	-2.070	2.070	0.067

"*" Denotes foreign region.

Table 5: Retirement model - Equilibrium values for each shock

		Output	Output*	Total Capital	Total Capital*	Cons.	Cons.*	Wealth	Wealth*	NFA	NFA*	LFPR	LFPR*	r
Mortality	Initial	2.112	2.080	15.950	15.703	1.939	1.937	15.557	16.090	-0.393	0.387	0.711	0.700	0.046
	Final	2.117	2.073	17.082	16.726	1.928	1.924	16.534	17.263	-0.549	0.537	0.688	0.674	0.043
Mortality & n	Initial	2.152	1.986	16.831	15.530	1.935	1.957	15.420	16.831	-1.410	1.301	0.711	0.656	0.045
	Final	2.142	1.922	18.588	16.675	1.932	1.941	16.693	18.375	-1.895	1.700	0.669	0.600	0.040
Dem. & SS	Initial	1.867	1.713	10.761	9.875	1.769	1.651	10.961	9.691	0.200	-0.184	0.727	0.667	0.061
	Final	1.824	1.619	11.557	10.261	1.711	1.612	11.119	10.650	-0.438	0.389	0.675	0.599	0.055

"*" Denotes foreign region.

Table 6: Comparison of NFA positions

Observed NFA position in hundreds of millions of 2010 US Dollars			
Country:	1980	2007	% Change
Canada	-2638.329	-226.7005	91.40742114
France	782.4547	3204.705	309.5706755
Germany	1081.599	9013.133	733.3155818
Japan	302.6127	22930.07	7477.365391
United Kingdom	850.539	-5999.803	-805.4118624
United States	5767.76	-24794.17	-529.8752029

Data from Lane, Milessi, and Ferreti (2007).

Table 7: 2000 wealth Gini coefficient by country

Country:	Wealth Gini Estimate:
Canada	0.688
France	0.73
Germany	0.667
Japan	0.547
United Kingdom	0.697
United States	0.801

Data retrieved from: Davies et al. (2007)

Table 8: Cross-country employment characteristics

	Age Dependency Ratio		Effective Retirement Age		Gross Replacement Rates %	
Country	1980	2010	1980	2010	1980	2010
Canada	13.8	20.4	64.9	63.4	34.0	44.0
France	21.8	26.4	63.5	59.4	66.0	49.0
Germany	23.9	31.3	60.0	62.0	49.0	43.2
Japan	13.4	36.0	70.7	70.1	54.0	35.7
United Kingdom	23.3	24.5	66.0	64.1	43.0	31.9
United States	17.2	19.4	66.4	65.5	44.0	38.7

Data retrieved from the World Bank World Development Indicators.
2010 replacement data from www.oecd.org.
1980 replacement data from Aldrich (1982).

Table 9: Productivity and labor share parameters

	TFP		Total labor share**	
	1980	2010	1980	2010
Canada	1	0.929	0.543	0.504
France	1	0.963	0.56	0.533
Germany	1	0.894	0.621	0.567
Japan	1	1.08	0.543	0.506
UK	1	1.31	0.536	0.543
US	1	1.34	0.568	0.533

*TFP values from the Penn World Table 8.1

**Share parameters from Nieman (2013)

Table 10: Labor force participation rates

Country:	1990	2010
Canada	76.0	71.5
France	65.2	62.0
Germany	69.6	66.3
Japan	77.2	71.6
UK	74.7	68.6
US	75.4	69.8

Data from the World Bank.

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Appendices

A.1 Baseline Model - Dynamic equation system as a function of domestic capital:

Noting from the no arbitrage condition that $r(t) = r^*(t)$ and using the fact that

$$f'(k) = f^{*\prime}(k^*) = r(t) \tag{13.1}$$

which using the explicit production function gives the capital relation:

$$k^* = \left(\frac{\alpha A}{\alpha^* A^*} k^{\alpha-1} \right)^{\frac{1}{\alpha^*-1}} \tag{13.2}$$

and

$$f''(k)\dot{k} = f^{*\prime\prime}(k^*)\dot{k}^* \tag{13.3}$$

which gives us:

$$\dot{k}^* = \left(\frac{\alpha A}{\alpha^* A^*} \right)^{\frac{1}{\alpha^*-1}} \left(\frac{\alpha - 1}{\alpha^* - 1} \right) k^{\frac{\alpha-\alpha^*}{\alpha^*-1}} \dot{k} \tag{13.4}$$

$$\dot{c}(t) = \varphi \frac{H(t)}{\Delta(t)} + \left(\sigma \left[\alpha A k(t)^{\alpha-1} - \rho \right] - n - \mu_c (t - \upsilon_1) \right) c(t) \tag{13.5}$$

$$\dot{c}^*(t) = \varphi^* \frac{H^*(t)}{\Delta^*(t)} + \left(\sigma^* \left[\alpha A k(t)^{\alpha-1} - \rho^* \right] - n^* - \mu_c^* (t - \upsilon_1^*) \right) c^*(t) \tag{13.6}$$

$$\dot{H}(t) = -(1-\alpha) A k(t)^{\alpha} + \left[\alpha A k(t)^{\alpha-1} + \mu_H (\tau_1 - t) \right] H(t) \tag{13.7}$$

$$\dot{H}^*(t) = -(1-\alpha^*) \left(\frac{\alpha A}{\alpha^* A^{*\frac{1}{\alpha^*}}} \right)^{\frac{\alpha^*}{\alpha^*-1}} k^{\frac{\alpha^*(\alpha-1)}{\alpha^*-1}} + \left[\alpha A k(t)^{\alpha-1} + \mu_H^* (\tau_1^* - t) \right] H^*(t) \tag{13.8}$$

$$\dot{\Delta}(t) = -1 + \left[(1-\sigma) \alpha A k(t)^{\alpha-1} + \sigma \rho + \mu_\Delta (\tau_2 - t) \right] \Delta(t) \tag{13.9}$$

$$\dot{\Delta}^*(t) = -1 + \left[(1-\sigma^*) \alpha A k(t)^{\alpha-1} + \sigma^* \rho^* + \mu_\Delta^* (\tau_2^* - t) \right] \Delta^*(t) \tag{13.10}$$

$$\dot{W}(t) = \left(\alpha A k(t)^{\alpha-1} - n \right) W(t) + (1-\alpha) A k(t)^{\alpha} - c(t) \tag{13.11}$$

$$\dot{W}^*(t) = \left(\alpha A k(t)^{\alpha-1} - n^* \right) W^*(t) + (1-\alpha^*) \left(\frac{\alpha A}{\alpha^* A^{*\frac{1}{\alpha^*}}} \right)^{\frac{\alpha^*}{\alpha^*-1}} k^{\frac{\alpha^*(\alpha-1)}{\alpha^*-1}} - c^*(t) \tag{13.12}$$

$$\dot{k}(t) = \chi \left(A k(t)^{\alpha} + A^* \left(\frac{\alpha A}{\alpha^* A^*} k^{\alpha-1} \right)^{\frac{\alpha^*}{\alpha^*-1}} - nW(t) - n^* W^*(t) - c(t) - c^*(t) \right) \tag{13.13}$$

A.1.1 Linearized Matrix of Baseline Model:

Where Ω is:

$$
\begin{pmatrix}
-\frac{\varphi\tilde{H}}{\tilde{\Delta}\tilde{c}} & 0 & \frac{\varphi}{\tilde{\Delta}} & 0 & -\frac{\varphi\tilde{H}}{\tilde{\Delta}^2} & 0 & 0 & 0 & \sigma(\alpha-1)\tilde{c}\frac{\tilde{r}}{\tilde{k}} \\
0 & -\frac{\varphi^*\tilde{H}^*}{\tilde{\Delta}^*\tilde{c}^*} & 0 & \frac{\varphi^*}{\tilde{\Delta}^*} & 0 & -\frac{\varphi^*\tilde{H}^*}{\tilde{\Delta}^{*2}} & 0 & 0 & \sigma^*(\alpha-1)\tilde{c}^*\frac{\tilde{r}}{\tilde{k}} \\
0 & 0 & \frac{\tilde{w}}{\tilde{H}} & 0 & 0 & 0 & 0 & 0 & (\alpha-1)\frac{\tilde{r}\tilde{H}}{\tilde{k}}-\alpha\frac{\tilde{w}}{\tilde{k}} \\
0 & 0 & 0 & \frac{\tilde{w}}{\tilde{H}^*}\left(\frac{A}{A^*}\right)^{\frac{1}{\alpha-1}} & 0 & 0 & 0 & 0 & (\alpha-1)\frac{\tilde{r}\tilde{H}^*}{\tilde{k}}-\frac{\alpha^*(\alpha-1)}{\alpha^*-1}\frac{\tilde{w}^*}{\tilde{k}} \\
0 & 0 & 0 & 0 & \frac{1}{\tilde{\Delta}} & 0 & 0 & 0 & (1-\sigma)(\alpha-1)\frac{\tilde{r}}{\tilde{k}}\tilde{\Delta} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\tilde{\Delta}^*} & 0 & 0 & (1-\sigma^*)(\alpha-1)\frac{\tilde{r}}{\tilde{k}}\tilde{\Delta}^* \\
-1 & 0 & 0 & 0 & 0 & 0 & \tilde{r}-n & 0 & (\alpha-1)\frac{\tilde{r}}{\tilde{k}}\tilde{W}+\alpha\frac{\tilde{w}}{\tilde{k}} \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & \tilde{r}-n^* & (\alpha-1)\frac{\tilde{r}}{\tilde{k}}\tilde{W}^*+\frac{\alpha^*(\alpha-1)}{\alpha^*-1}\frac{\tilde{w}^*}{\tilde{k}} \\
-\chi & -\chi & 0 & 0 & 0 & 0 & -n\chi & -n^*\chi & \chi\left(\tilde{r}+\frac{\alpha^*(\alpha-1)}{\alpha^*-1}\frac{f^*(\tilde{k})}{\tilde{k}}\right)
\end{pmatrix}
$$

(13.14)

Where:

$$\chi=\left[\left(\frac{\alpha A}{\alpha^* A^*}\right)^{\frac{1}{\alpha^*-1}}\left(\frac{\alpha-1}{\alpha^*-1}\right)k^{\frac{\alpha-\alpha^*}{\alpha^*-1}}+1\right]^{-1} \qquad (13.15)$$

A.2 Retirement Model – Derivation of Equation (11.10)

$$H(v,t) \equiv \int_t^{v+R} (1-\tau_s) w(\tau) e^{-R(t,\tau)-M(\tau-v)+M(t-v)} d\tau + \int_{v+R}^{v+D} B(\tau) e^{-R(t,\tau)-M(\tau-v)+M(t-v)} d\tau \quad (14.1)$$

$$H(t) = H(t,t) = \int_t^{t+R} (1-\tau_s) w(\tau) e^{-R(t,\tau)-M(\tau-t)} d\tau + \int_{t+R}^{t+D} B(\tau) e^{-R(t,\tau)-M(\tau-t)} d\tau$$

$$\dot{H}(t) = (1-\tau_s) w(t+R) e^{-R(t,t+R)-M(R)} - (1-\tau_s) w(t) + \int_t^{t+R} \left[r(t) + \mu_{H1} \right] (1-\tau_s) w(\tau) e^{-R(t,\tau)-M(\tau-t)} d\tau$$
$$-\beta w(t+R) e^{-R(t,t+R)-M(R)} + \int_{t+R}^{t+D} \left[r(t) + \mu_{H2} \right] \beta w(\tau) e^{-R(t,\tau)-M(\tau-t)} d\tau$$

$$\dot{H}(t) = \left[(1-\tau_s) - \beta \right] w(t+R) e^{-R(t,t+R)-M(R)} - (1-\tau_s) w(t) + \left[r(t) + \mu_{H1} \right] H(t)$$
$$-\left[(1-\tau_s) - \beta \right] \left[\mu_{H2} - \mu_{H1} \right] \int_{t+R}^{t+D} w(\tau) e^{-R(t,\tau)-M(\tau-t)} d\tau$$

Given that the difference of the mortality is approximately zero, the last term drops out.

Now employing the log-linear procedure of Mierau and Turnovsky (2015):

$$Z(t+R) \equiv w(t+R) e^{-R(t,t+R)-M(R)}$$

$$\ln Z(t+R) = \ln w(t+R) - \int_t^{t+R} r(s) ds - \int_0^R \mu(s) ds \quad (14.2)$$

$$\ln Z(t+R) \approx \ln Z(t) + \left. \frac{d \ln Z(t+R)}{dR} \right|_{R=0} R$$

This is the extrapolation from age 0 to age Rand then discounting it back to age 0.

Applying this method to (14.2) generates:

$$w(t+R) e^{-R(t,t+R)-M(R)} \approx w(t) e^{\left[\frac{\dot{w}(t)}{w(t)} - r(t) \right] R} \quad (14.3)$$

A.3 Retirement Model – Per-worker Dynamic Equations and linearization:

$$\dot{c}(t)=\frac{\varphi}{l}\frac{H(t)}{\Delta(t)}+\left(\sigma\left[r(t)-\rho\right]-n-\mu_c\right)c(t) \tag{15.1}$$

$$\dot{c}^*(t)=\frac{\varphi^*}{l^*}\frac{H^*(t)}{\Delta^*(t)}+\left(\sigma^*\left[r(t)-\rho^*\right]-n^*-\mu_c^*\right)c^*(t) \tag{15.2}$$

$$\dot{H}(t)=-(1-\tau_s)w(t)+\left[r(t)+\mu_H(\tau-t)\right]H(t)-\left[\beta-(1-\tau_s)\right]w(t)e^{\left[\frac{\dot{w}(t)}{w(t)}-r(t)\right]R} \tag{15.3}$$

$$\dot{H}^*(t)=-(1-\tau_s^*)w^*(t)+\left[r(t)+\mu_H^*(\tau-t)\right]H^*(t)-\left[\beta^*-(1-\tau_s^*)\right]w(t)e^{\left[\frac{\dot{w}^*(t)}{w^*(t)}-r(t)\right]R^*} \tag{15.4}$$

$$\dot{\Delta}(t)=-1+\left[(1-\sigma)r(t)+\sigma\rho+\mu_\Delta(\tau-t)\right]\Delta(t) \tag{15.5}$$

$$\dot{\Delta}^*(t)=-1+\left[(1-\sigma^*)r(t)+\sigma^*\rho^*+\mu_\Delta^*(\tau-t)\right]\Delta^*(t) \tag{15.6}$$

$$\dot{W}(t)=\left(r(t)-n\right)W(t)+w(t)-c(t) \tag{15.7}$$

$$\dot{W}^*(t)=\left(r(t)-n^*\right)W^*(t)+w^*(t)-c^*(t) \tag{15.8}$$

$$\dot{k}(t)+\dot{k}^*(t)=f\left(k(t)\right)+f^*\left(k^*(t)\right)-nW(t)-n^*W^*(t)-c(t)-c^*(t) \tag{15.9}$$

Linearization:

$$\begin{pmatrix}\dot{c}(t)\\ \dot{c}^*(t)\\ \dot{H}(t)\\ \dot{H}^*(t)\\ \dot{\Delta}(t)\\ \dot{\Delta}^*(t)\\ \dot{W}(t)\\ \dot{W}^*(t)\\ \dot{k}(t)\end{pmatrix}=\Omega\begin{pmatrix}c(t)-\tilde{c}\\ c^*(t)-\tilde{c}^*\\ H(t)-\tilde{H}\\ H^*(t)-\tilde{H}^*\\ \Delta(t)-\tilde{\Delta}\\ \Delta^*(t)-\tilde{\Delta}^*\\ W(t)-\tilde{W}\\ W^*(t)-\tilde{W}^*\\ k(t)-\tilde{k}\end{pmatrix} \tag{16.1}$$

Where Ω is:

$$\begin{pmatrix}
-\frac{\varphi\tilde{H}}{\tilde{\Delta}\tilde{c}\tilde{l}} & 0 & \frac{\varphi}{\tilde{\Delta}\tilde{l}} & 0 & -\frac{\varphi\tilde{H}}{\tilde{l}\,\tilde{\Delta}^{2}} & 0 & 0 & 0 & \sigma(\alpha-1)\tilde{c}\,\frac{\tilde{r}}{\tilde{k}} \\
0 & -\frac{\varphi^{*}\tilde{H}^{*}}{\tilde{l}\,\tilde{\Delta}^{*}\tilde{c}^{*}} & 0 & \frac{\varphi^{*}}{\tilde{l}\,\tilde{\Delta}^{*}} & 0 & -\frac{\varphi^{*}\tilde{H}^{*}}{\tilde{l}\,\tilde{\Delta}^{*2}} & 0 & 0 & \sigma^{*}(\alpha-1)\tilde{c}^{*}\,\frac{\tilde{r}}{\tilde{k}} \\
L_{31} & L_{32} & L_{33} & 0 & 0 & 0 & L_{37} & L_{38} & L_{39} \\
L_{41} & L_{42} & 0 & L_{44} & 0 & 0 & L_{47} & L_{48} & L_{49} \\
0 & 0 & 0 & 0 & \frac{1}{\tilde{\Delta}} & 0 & 0 & 0 & (1-\sigma)(\alpha-1)\frac{\tilde{r}}{\tilde{k}}\tilde{\Delta} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\tilde{\Delta}^{*}} & 0 & 0 & (1-\sigma^{*})(\alpha-1)\frac{\tilde{r}}{\tilde{k}}\tilde{\Delta}^{*} \\
-1 & 0 & 0 & 0 & 0 & 0 & \tilde{r}-n & 0 & (\alpha-1)\frac{\tilde{r}}{\tilde{k}}\tilde{W}+\alpha\frac{\tilde{w}}{\tilde{k}} \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & \tilde{r}-n^{*} & (\alpha-1)\frac{\tilde{r}}{\tilde{k}}\tilde{W}^{*}+\alpha\left(\frac{A}{A^{*}}\right)^{\frac{1}{\alpha-1}}\frac{\tilde{w}}{\tilde{k}} \\
-\chi & -\chi & 0 & 0 & 0 & 0 & -n\chi & -n^{*}\chi & \tilde{r}
\end{pmatrix}$$

(16.2)

$$L_{31}=\frac{1}{\tilde{k}}\left[\beta-\left(1-\tau_{s}\right)\right]\tilde{w}\alpha\chi Re^{-\tilde{r}R} \tag{16.3}$$

$$L_{32}=L_{31} \tag{16.4}$$

$$L_{33}=\frac{\tilde{w}}{\tilde{H}}\left(\left(1-\tau_{s}\right)\left(1-e^{-\tilde{r}R}\right)+\beta e^{-\tilde{r}R}\right) \tag{16.5}$$

$$L_{37}=\frac{1}{\tilde{k}}\left[\beta-\left(1-\tau_{s}\right)\right]\tilde{w}\alpha\chi Rne^{-\tilde{r}R} \tag{16.6}$$

$$L_{38}=L_{37}\frac{n^{*}}{n} \tag{16.7}$$

$$L_{39}=\frac{1}{\tilde{k}}\left(\left(\alpha-1\right)\tilde{r}\tilde{H}-\left(1-\tau_{s}\right)\alpha\tilde{w}-\left[\beta-\left(1-\tau_{s}\right)\right]\tilde{w}e^{-\tilde{r}R}\left(\alpha+\frac{\tilde{r}R}{\tilde{k}}\right)\right) \tag{16.8}$$

$$L_{41}=\left(\frac{A}{A^{*}}\right)^{\frac{1}{\alpha-1}}\frac{1}{\tilde{k}}\left[\beta^{*}-\left(1-\tau_{s}^{*}\right)\right]\tilde{w}\alpha\chi R^{*}e^{-\tilde{r}R^{*}} \tag{16.9}$$

$$L_{42}=L_{41} \tag{16.10}$$

$$L_{44}=\left(\frac{A}{A^{*}}\right)^{\frac{1}{\alpha-1}}\frac{\tilde{w}}{\tilde{H}^{*}}\left(\left(1-\tau_{s}^{*}\right)\left(1-e^{-\tilde{r}R^{*}}\right)+\beta^{*}e^{-\tilde{r}R^{*}}\right) \tag{16.11}$$

$$L_{47}=\left(\frac{A}{A^{*}}\right)^{\frac{1}{\alpha-1}}\frac{1}{\tilde{k}}\left[\beta^{*}-\left(1-\tau_{s}^{*}\right)\right]\tilde{w}\alpha\chi R^{*}ne^{-\tilde{r}R^{*}} \tag{16.12}$$

$$L_{48} = L_{47} \frac{n^*}{n} \tag{16.13}$$

$$L_{49} = \frac{1}{\tilde{k}} \left((\alpha - 1) \tilde{r} \tilde{H}^* - \left(\frac{A}{A^*} \right)^{\frac{1}{\alpha - 1}} \left(1 - \tau_s^* \right) \alpha \tilde{w} - \left[\beta^* - \left(1 - \tau_s^* \right) \right] \left(\frac{A}{A^*} \right)^{\frac{1}{\alpha - 1}} \tilde{w} e^{-\tilde{r} R^*} \left(\alpha + \frac{\tilde{r} R^*}{\tilde{k}} \right) \right) \tag{16.14}$$