On Aggregate Fluctuations, Systemic Risk, and the Covariance of Firm-Level Activity

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Abstract

I study firm-level covariance of productivity, sales, and profit for Compustat firms over the last half-century, in order to understand the sources of aggregate variance and systemic risk for these firms. I find that firm-level covariance explains most of aggregate variance in Compustat, that high-productivity firms covary more with aggregate productivity, sales, and profit growth, but less per dollar of market value. I develop a theory based on diversification of business lines to explain these facts. I derive propositions that characterize endogenous first and second moments of firm and aggregate productivity, and relate firms’ expected stock returns to their endogenous firm-aggregate productivity covariances. As a plausibility check, I run regressions on the number of business segments Compustat firms report and find tentative support for the model’s predictions.

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1 Introduction

I drive to work—it’s faster, but driving is risky. My point is this: some risks I choose, they are not imposed upon me. Firms also choose risks: they choose their production methods, they choose their product lines, and when they do, their decisions expose them to technology- and product-specific risks. I argue that when many firms choose similar risks, perhaps drawn to similar rewards, their economic fortunes all hang on the same events, and rise and fall together. This comovement of individual outcomes creates aggregate fluctuations, the chosen risks become systemic. To motivate this interpretation, I document four patterns in the comovement of firm-level activity for a large panel of publicly-traded firms in the United States over the last half-century, and develop a model economy that produces the patterns endogenously by the mechanism just described. My contributions build on recent work on endogenous uncertainty in macroeconomic models, microeconomic origins of aggregate fluctuations, and financial risk in production economies.

The motivating evidence on firm-level covariance is based on a well-known decomposition: aggregate variance equals the sum of individual variances and pairwise covariances. Details of the decomposition depend on the aggregate in question—whether its elements are additively separable, whether they are growth rates or levels, whether entry and exit occur—but often a simple mathematical identity or approximation holds, and is useful for thinking about the sources of aggregate variance. I display the simplest version of the decomposition directly in the second line of equation (1), because of its central role here. For aggregate $X = \sum_{\Omega} x_\omega$, where $\omega$, $\omega'$ index firms in set $\Omega$, and $x_\omega$ is a firm variable:

\[
\begin{align*}
\text{Var}(X) &= \text{Var}\left(\sum_{\Omega} x_\omega\right) \\
&= \sum_{\Omega} \text{Var}(x_\omega) + \sum_{\Omega} \sum_{\Omega \setminus \{\omega\}} \text{Cov}(x_\omega, x_{\omega'}) \\
&= \sum_{\Omega} \text{Cov}(x_\omega, X).
\end{align*}
\]

The third line in (1) is also useful. It says that a firm’s covariance with the aggregate is simultaneously its contribution to aggregate variance.

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Now consider productivity, sales, and profit growth for Compustat firms: on average over the last half-century, covariances between firms accounted for about 85% of variance in aggregate growth rates for these variables. Firms in the high-productivity decile contributed on average over six times as much variance to aggregate growth rates as the average firm in the sample. High-productivity firms are driving aggregate fluctuations, and doing so through covariance with other firms. There are about 6,500 distinct firms in the Compustat sample over this period, coming from nearly all industries, and producing goods equal in value to about 20% of U.S. gross domestic product each year. Covariance matters for the aggregate fluctuations of these firms.

It also matters for their risk. Markowitz made this point in 1952 for portfolio returns, using a version of the above decomposition, but with weights. Covariance risk underlies the portfolio-based capital asset pricing models of Sharpe (1964), Lintner (1965), and Mossin (1966), and also Breeden (1979)'s consumption-based model. Mine is a productivity-based model in which a firm's risk depends, predominantly, on the covariance between the firm's productivity and aggregate productivity, relative to the firm's market value—the last term on the right-hand side my model's equation for expected excess returns:

\[
E [r_t(\omega) - r_{f,t}] \approx \zeta_1 \frac{\mu(\omega)}{V_t(\omega)} + \zeta_2 \frac{\sigma_{\omega_t}(\omega)}{V_t(\omega)},
\]

where the left-hand side gives expected excess returns for firm \( \omega \), \( V_t(\omega) \) the firm's market value, \( \mu(\omega) \) the firm's expected productivity, and \( \sigma_{\omega_t}(\omega) \) is firm-aggregate productivity covariance. The ratio covariance-over-value may help explain why investors view high-productivity firms as less risky, as İmrohoroğlu and Tüzel (2014) have recently pointed out in stock returns data, despite these firms driving aggregate fluctuations. İmrohoroğlu and Tüzel estimate that high-productivity firms have on average 7.35% lower annual expected returns. Equation (2) attributes that difference in returns, roughly, to the ratio of covariance in economic activity to value of economic activity. Evidence from the Compustat data also points in this direction: while the firm-aggregate covariances of high-productivity firms are six times those of the average firm, this multiple falls to less than one third, on a dollar-for-dollar basis, after dividing by market value. This is true of firm-aggregate covariance in productivity, sales, and profit, as Figure 1 illustrates. Evidently, high-productivity
firms are doing at least some business that is profitable but weakly correlated with the business of
other firms. This diversification of business activity is also suggested by direct measurement in the
Compustat segment data, where high-productivity firms operate on average 2.6 segments, against
1.5 for low-productivity firms (see Table 1). In theory, a risk-averse investor should be willing to
accept lower returns from better-diversified firms, because these firms’ activities covary less with
aggregate activity, relative to the profit investors expect to receive. In the classical capital asset
pricing models, the relevant covariance is between firm and portfolio or market returns, and in
consumption-based models it is between firm returns and marginal utility of consumption. Here,
the relevant covariance is between firm and aggregate productivity, and it arises endogenously for
each firm.

Figure 1 summarizes the empirical evidence on firm-level covariance that motivates my theory.
To highlight the pervasiveness of covariance in all stages of value creation, the figure illustrates the
evidence for growth rates of three variables: total factor productivity estimated by the Olley and
Pakes (1996) method, net sales, and operating income before depreciation, which I refer to simply
as profit. Panel (a) shows the fraction of variance in aggregate growth due to pairwise covariance in
firm-level growth. In most years, pairwise covariances account for 80–90% of aggregate variance.
Panel (b) illustrates firm-aggregate covariance for firms sorted into productivity deciles. Recall
that firm-aggregate covariance measures an individual firm’s contribution to aggregate variance.
High-productivity firms contribute over six times what the average firm contributes. Panel (c)
illustrates firm-aggregate covariance relative to market value. Interpret this statistic as the systemic
risk the firm poses investors per dollar of investment. High-productivity firms expose investors to
one-third the amount of risk the average firm does. The firm-level evidence on covariance provides a
useful yardstick to measure the success of economic models explaining cross-sectional stock returns
or microeconomic origins of aggregate fluctuations. My first contribution is to carefully document
this evidence.

Because firm-level covariance appears to drive systemic risk and aggregate fluctuations in the

1Authors in the finance literature have primarily focused on the impact of firm diversification on firm value.
Villalonga (2004) finds a value premium on diversified firms, consistent with the theory here. But here, the emphasis
is on diversification and stock returns. In this context, Wang (2012) finds that less-diversified firms pay higher returns,
also consistent with the theory presented here. I discuss this further in section 2.
data, a single model with an empirically plausible firm-level covariance structure can contribute to explanations of both systemic risk and aggregate fluctuations. My second contribution is along these lines. I construct a DSGE model in which firm-level covariance in economic activity arises endogenously. The model produces aggregate fluctuations and cross-sectional patterns in firm-level risk that are consistent with the empirical patterns. I endogenize covariance by allowing heterogeneous firms to choose risky business lines for themselves, from a menu that I specify exogenously. When firms choose similar business lines, their productivities covary. Mathematically, covariance in the model has a simple factor structure that is flexible and tractable. The mechanism relies on high-productivity firms choosing to operate a greater number of business lines, just as they do in the data. My model predicts that firms with more business lines have higher firm-aggregate covariance, and lower firm-aggregate covariance-over-value. The motivating evidence in Figure 1 demonstrates the empirical plausibility of these model predictions, and regressions controlling for alternative explanations provide formal inference. I count each firm’s reported business lines using Compustat segment data, regress covariance on this measure and a set of control variables, and find tentative support for the business line diversification view. My theory also predicts that firms with similar productivities have higher correlations with each other, and this pattern is weakly visible in the data and illustrated in panel (d) of Figure 1.

The rest of the paper is organized as follows: In section 2 I discuss my work in the context of existing literature. In section 3 I introduce a formal model based on business-line diversification. In section 4, I derive propositions that highlight the model’s main mechanism, and demonstrate the model’s consistency with the motivating empirical evidence. In section 5, I provide details on productivity estimation and firm-level covariance calculations, and run regressions to check the plausibility of the business-line diversification hypothesis. I conclude in section 6.

## 2 Literature

My work relates to recent work on endogenous uncertainty in macroeconomic models, microeconomic origins of aggregate fluctuations, financial risk in production economies, business cycles in the firm
(a) Sum of Pairwise Covariance
Sum of pairwise firm-level growth-rate covariances, divided by aggregate growth rate variance.

(b) Firm-Agg Covariance
Covariance between firm and aggregate growth rates, relative to the cross-sectional average, averaged within decile and over time.

(c) Firm-Agg Cov-over-Value
Covariance between firm and aggregate growth rates over end-of-window market value, relative to the cross-sectional average, averaged within decile and over time.

(d) Cross-Decile Correlation
Averages of yearly pairwise firm correlations for firm pairs in decile-square sets $\Omega_i \times \Omega_j$, \forall i, j = 1, 2, ..., 10, where $\Omega_i$ is the set of decile i firms.

Figure 1 – Evidence from Compustat Firm-Level Covariances. Compustat Annual Fundamentals, 1961-2015. Firm-level covariance statistics are computed in 6-year rolling windows, and firms are grouped into deciles according to moving averages of their estimated total factor productivities. The sample includes all Compustat firms with no missing values in a given 6-year window, excluding financial and utilities firms, firms with large merges, and the smallest 10% of firms by market value. Details of the dataset and computations used to produce this figure are given in section 5.
cross section, and corporate diversification. A notable feature of the framework I propose is that it accords, at least qualitatively, with this number of disparate areas of economic research. For example, the model in section 3 endogenously generates aggregate uncertainty from microeconomic shocks, captures features of the cross-section of stock returns, matches a host of stylized facts related to cross-sectional volatility and co-movement, and captures empirical regularities related to the number of technologies, product lines, and business segments firms operate. Yet the model’s mechanism is simple: firms choose their technologies. The following paragraphs relate my work to contributions in these fields.

**Endogenous uncertainty in macroeconomic models.** The financial crisis of 2007 led to calls for macroeconomic models in which aggregate fluctuations arise endogenously. Romer (2016) complains about models in which aggregate fluctuations “are not influenced by the action that any person takes.” Stiglitz (2011) questions the relevance of real business cycle models for work on recessions, because the models presume “that the origin of fluctuations [is] exogenous.” These criticisms apply only partially to the class of models to which mine belongs: production economies with firm heterogeneity and random productivity. In many models within this class, exogenous productivity shocks at the firm level propagate endogenously from firms to aggregate variables. Because the propagation is endogenous, the aggregate fluctuations are endogenous. Dutta and Polemarchakis (1992) offer an early example of this type of propagation, Acemoglu et al. (2012) and Gabaix (2011) are recent example, and I review more of this literature below.

Still, within this class of models, productivity shocks are exogenous at the firm level, so at the firm level the Stiglitz-Romer critique is valid. In my model, which is a new member of this class, individual firms choose their production technologies, and their choices determine the probability distribution of their productivity shocks. When many firms choose the same technology, shocks to that technology have a larger aggregate impact. In this sense, I provide a framework for addressing the Stiglitz-Romer critique in production economies with heterogeneous firms and random productivity.

Danielsson and Shin (2003) provide an early example of endogenous risk that pre-dates the crisis and the Stiglitz-Romer critique, but theirs is a partial-equilibrium asset pricing model that belongs to a different class.
Origins of aggregate fluctuations. Early work in this area argued that idiosyncratic uncertainty at the individual level was not a plausible source of aggregate uncertainty. For example, in work on general equilibrium theory, Malinvaud (1972) and Malinvaud (1973) considered large endowment economies where equilibrium prices are certain despite agents facing idiosyncratic shocks. In the growth accounting literature, Hulten (1978) showed that, to a first-order, the aggregate impact of sectoral productivity shocks is proportional to sectors’ sales shares, and therefore likely to be small in large, efficient economies. The same logic, according to later arguments, applies to firm-level shocks.

Recently, authors have challenged the early work. Baqaee and Farhi (2017) find that idiosyncratic shocks have much larger aggregate effects when Hulten’s approximation is taken to a second order. Gabaix (2011) argues that idiosyncratic shocks matter more in economies dominated by large firms (more “granular” economies, in his language), and Carvalho and Gabaix (2013) use a granular model economy to explain movements in U.S. GDP volatility. Acemoglu et al. show how input-output networks can magnify idiosyncratic shocks, focusing on sectoral shocks, and Kelly et al. (2013) develop a model in which granular effects at the firm level lead to sectoral shocks that propagate through networks and generate aggregate fluctuations.

My work differs from this recent work in one important way: I emphasize covariance in firm-level shocks, rather than idiosyncratic shocks, as the crucial firm-level source of aggregate fluctuations. My focus on covariance is motivated by the Compustat evidence, where firm-level covariance appears to explain 80–90% of aggregate variance, even for productivity growth, which is usually modeled as idiosyncratic. While none of the recent work places quite the same emphasis on covariance that I do, Baqaee and Farhi (2017) do consider the case of correlated shocks in their second-order approximations and find that correlated shocks can have large aggregate effects. Earlier researchers also briefly considered correlated shocks: Malinvaud (1972) and Dutta and Polemarchakis (1992) consider cases where microeconomic shocks covary, but where covariance decays in the distance between the shocks along an arbitrary index dimension. They find in these cases that covariance in the small does not lead to uncertainty in the large if the decay is rapid enough. In my model, covariance does not decay rapidly; this is because, roughly, some technologies are used by nearly all
firms and are therefore highly systemic. This statement is true even for a continuum of technologies and a continuum of firms, and I provide formal justification for it in the propositions in section 4.

Finally, it is worth emphasizing that a covariance-based explanation for aggregate fluctuations needn’t compete directly with explanations based on idiosyncratic shocks: It is perfectly possible for firm-level covariance to explain a large fraction of aggregate variance in a typical period, but a small fraction of the changes in aggregate variance from period to period. The granularity view seems well suited to explain the period-to-period changes. Carvalho and Gabaix (2013) study swings in U.S. GDP volatility since 1960, and find that changes in GDP volatility are well explained by changes in firm-level volatility. This view is consistent with the motivating evidence in panel (a) of Figure 1, where occasional downward spikes in the figure during crisis years represent a relative increase in the importance of firm-level variance for explaining aggregate fluctuations.

Financial risk in production economies. Explaining cross-sectional differences in stock returns has long been of interest in finance. van Dijk (2011) reviews thirty years of research on differences related to firm size, and remarks on a trend toward modeling the cross-section of stock returns in general equilibrium production economies. This new research seeks an “economic theory that identifies the state variables that drive variation in returns related to firm size.”

The literature starts with Berk et al. (1999), who develop a partial equilibrium model in which firms invest in projects with uncertain cash flows and durations. The cash flows covary with an exogenous pricing kernel, and a firm’s set of active projects determines its risk. My work is closely related to Berk et al., with production technologies here playing a similar role to investment projects there. Our models differ in two important ways: first, in my work each technology’s covariance with aggregate productivity is endogenous—it depends on the number of firms that choose to operate the technology. Second, my pricing kernel is endogenous, and, through market clearing conditions, ultimately also depends on firm-level technology choices.

Gomes (2001) develops a general equilibrium model in which financing constraints generate an empirically plausible cross section of stock returns. More recent models have relied on convex capital adjustment costs to produce differences in returns, for example Gomes et al. (2003), İmrohoroğlu and Tüzel (2014), and Zhang (2017). Donangelo et al. (2017) propose a novel labor leverage mechanism.
I view my work as complementary to these lines. I emphasize business line diversification as a source of cross-sectional differences in stock returns. While the diversification argument is not new to the finance literature, my implementation in a general equilibrium production economy is new, to the best of my knowledge. I discuss the diversification literature further below.

Finally, the real business cycle literature also studies stock returns in production economies, but focuses mostly on the time series properties of returns. Examples include Rouwenhorst (1995), Jermann (1998), and Lettau (2003). In my work, as in the traditional real business cycle literature, fluctuations are driven by technology shocks. But in my work there is a continuum of technologies and firms solve a technology choice problem; these features provide a microeconomic foundation for the traditional aggregate technology shock, and, in particular, one that is consistent with the cross-sectional evidence on stock returns.

The cross section of firms over the business cycle. The motivating evidence in panels (b) and (c) of Figure 1 describes how firm-aggregate covariance and covariance-per-dollar vary in the productivity cross section of firms. That analysis differs from, but compliments, recent and earlier empirical work on the cross section of firms over the business cycle. The early work is most closely associated with Gertler and Gilchrist (1994), who find that small firms respond more than large firms to monetary policy events because, as they argue, small firms face greater credit market frictions. Chari et al. (2007), revisiting the Gertler and Gilchrist (1994) work, find in a longer time series that small and large firms respond in the same way to fluctuations in aggregate economic activity. Both studies use QFR manufacturing data. Other recent findings are mixed: Gourio (2007) finds that small firms’ profits are more procyclical. Crouzet and Mehrotra (2017) find that small firms are slightly more responsive, but only more so than the largest half-percent of firms, and not because of financial frictions. Moscarini and Postel-Vinay (2009) focus on employment and find in multiple datasets that large firms are more responsive.

I contribute an additional data point and a new perspective on this line of empirical work. My methodology differs, in that I use an aggregate variance decomposition to highlight the dual nature of firm-aggregate covariance: on the one hand, firm-aggregate covariance measures firm contributions to aggregate variance, on the other, it measures firm exposure to the business cycle. I find that
productivity, sales and profit growth rates at high-productivity firms covary more with aggregate growth rates, but less per dollar of market value. The interpretation is that high-productivity firms contribute more to aggregate variance, but are less exposed to business cycle risk.

Corporate diversification. The key mechanism in my model is business-line diversification, where a business line consists of a single technology and the consumption good it produces, and where high-productivity firms operate more business lines. Empirical evidence suggests that high-productivity firms are indeed better diversified, both in terms of product lines and production methods. Bernard et al. (2010) document the prevalence of multi-product firms in U.S. manufacturing, and in regressions they find a positive correlation between product adding and firm-level productivity. Broda and Weinstein (2010) analyze AC Nielsen bar code data and find that large firms sell a far greater number and variety of products at the upc, brand, module, and product group levels. Large firms are on average high-productivity firms, so this work also suggests greater product diversification at high-productivity firms. Evidence on technology use is scarce, but Dunne (1991) uses data from the U.S. Census Survey of Manufacturing Technology to estimate adoption probabilities for seventeen advanced technologies, and finds that large manufacturers were more likely than small to adopt each of the seventeen technologies in the survey.

In theory, firm diversification often entails technological change. For Ansoff (1957), firms diversify when they sell new products in new markets, and diversification typically requires “new skills, new techniques, and new facilities.” Frankel (1955) provides an early description of the costs and considerations associated with introducing new production methods alongside old, and Gort (1962) argues that diversifying firms typically enter fast growing industries with high rates of technological change. My model captures much of this technological diversification theory by allowing product diversification only through technology adoption, and by introducing a fixed cost for each technology that firms operate. In the name of simplicity, the model does not differentiate goods by industry (beyond consumption and capital) so it misses some of the richness of Gort’s theory. It also misses strategic aspects of technology adoption, as emphasized by Reinganum (1981) and Fudenberg et al. (1983), for example.

Gollop and Monahan (1991) survey a literature on index measures of firm diversification. The
best measures are sensitive to a firm’s product count, the distribution of its sales across products, and the similarity of the products. Unfortunately, Compustat segment data only allow for simple business line counts, a coarse measure of diversification with little cross-sectional variation. For example, high-productivity firms report on average 2.6 segments each, compared to 1.5 for low-productivity firms. Nevertheless, in section 5 I use the segment counts to test a prediction of my model: that firms with more business lines covary more with aggregates, but less per dollar of market value. Despite the coarseness of the diversification measure, the regressions provide some empirical support for the model predictions.

Finally, a large finance literature focuses on corporate diversification and firm value. Martin and Sayrak (2003) survey the literature and describe the prevailing view in the 1990s as one of diversified firms trading at discounts. With the availability of more granular data, this view is changing. For example, Villalonga (2004) compares Compustat data with U.S. Census data that allows for finer measures of diversification and finds that diversified firms trade at a premium relative to focused firms in the Census segment data but not in the coarser Compustat segment data. Wang (2012) looks at stock returns rather than firm value, and finds that diversified firms in Compustat have lower returns on average. He argues that diversification affects a firm’s systemic risk though its growth options. The predictions of my model are consistent with the recent work by Villalonga and Wang.

3 Model Primitives

I now construct a simple two-sector production economy that rationalizes the motivating evidence presented in the introduction. In the model, firms produce capital and consumption goods for a representative household. The capital good is produced by a representative firm, while differentiated consumption goods are produced by a continuum of monopolistically competitive firms. The latter firms each produce a number of different varieties using a number of different technologies. The technologies are non-rivalrous, so any number of firms can use the same technology, and the varieties are differentiated by technology and producer, so two firms using the same technology produce two
different varieties. Firms choose their technology sets from a continuum of available technologies, each technology is a distinct source of randomness, and technology is the only source of randomness in the model. The remaining primitive assumptions and equations of the model are given below, and propositions in section 4 explain the main mechanisms and highlight key results.

3.1 Consumption Goods Producers

Firms indexed by \( \omega \in \Omega = [0, 1] \) compete monopolistically for household consumption demand. They use multiple technologies and produce a differentiated variety of consumption good with each technology they use.

3.1.1 Production

To produce consumption goods, firms use technologies that combine labor and capital in a constant-returns-to-scale fashion. Each technology has its own random productivity multiplier, denoted \( z_{t,v} \). Technological productivity is the first of two productivity types in the model, and is the only source of randomness. The second productivity type is firm-specific and non-random, and denoted \( z(\omega) \). One interpretation is that the firm-specific productivity reflects management. Bloom et al. (2016) document a wide dispersion in management practices across firms, and find evidence that these differences explain some of the observed dispersion in total factor productivity across firms. I assume firm-specific productivity is Pareto distributed across firms, with shape parameter \( \kappa \) and scale parameter set to unity. I assume Cobb-Douglas production functions, and write firm \( \omega \)'s output \( y_{t,v}(\omega) \) of the variety created by technology \( v \) at time \( t \) as

\[
y_{t,v}(\omega) = z(\omega) z_{t,v} [k_{t,v}(\omega)]^\alpha [l_{t,v}(\omega)]^{1-\alpha},
\]

(3)

where parameter \( \alpha \) controls the cost share attributed to the capital \( k_{t,v}(\omega) \) and the labor \( l_{t,v}(\omega) \) the firm uses to produce the good.

Finally, note that capital is homogeneous in this set-up, in contrast to traditional vintage capital models. Firms can move capital freely across technologies and combine it with labor in varying
proportions. This assumption is analytically convenient, and increasingly plausible economically, in light of the increasing flexibility of manufacturing systems observed by Milgrom and Roberts (1990).²

### 3.1.2 Profit Maximization

Profits and prices are expressed in units of an aggregate consumption basket specified in subsection 3.2. Firms take the wage $w_t$ and the capital rental rate $r_t$ as given, but act as monopolists in each of their differentiated varieties, setting prices to maximize profits. Denote by $p_{t,v}(\omega)$ the price that firm $\omega$ sets in period $t$ for the variety it produces with technology $v$, and write the firm’s gross profit from producing $y_{t,v}(\omega)$ units of the variety as

$$\pi_{t,v}(\omega) = p_{t,v}(\omega)y_{t,v}(\omega) - r_t k_{t,v}(\omega) - w_t l_{t,v}(\omega). \quad (4)$$

A firm’s total gross profit $\Pi_t(\omega)$ equals the sum of gross profits from each of its varieties:

$$\Pi_t(\omega) = \int_{\mathcal{V}(\omega)} \pi_{t,v}(\omega) \lambda(dv).$$

Firms are owned by the representative household, so they use the household’s stochastic discount factor to discount expected future profits. They maximize value by choosing optimal factor inputs and prices for each differentiated good, subject to the production function given by equation (3), and subject to household demand given by equation (14). Let $\mathcal{V}(\omega)$ represent the set of technologies that firm $\omega$ uses, and write the firm’s decision problem as

$$\max \left\{ k_{t,v}(\omega), l_{t,v}(\omega), p_{t,v}(\omega) \right\}_{v \in \mathcal{V}(\omega)} V_t(\omega) = E_t \left[ \sum_{s=t+1}^{\infty} m_{t,s} \Pi_s(\omega) \right] \quad (5)$$

s.t. (3) and (14) $\forall \ v \in \mathcal{V}(\omega)$.

²In a future version I intend to extend the model by indexing capital according to vintage and restricting the use of capital to technologies of corresponding vintage. The extension would differ from the putty-clay assumption made in many vintage capital models, in that it would place no restriction on the proportions in which inputs are combined in production. It would, however, bring the model closer to vintage capital models in the style of Solow (1960), which tend to allow for easier aggregation. See Johansen (1959) for an early putty-clay model, or Boucekkine et al. (2011) for a recent survey of vintage capital models of both varieties.
3.1.3 Technology Choice

Firms choose their technology sets, denoted $\mathcal{V}(\omega)$, from a fixed set $\mathcal{V} = [v, \infty) \subset \mathbb{R}^+$ of available technologies, and make their choices one period in advance. There is no cost to adopting or abandoning a technology, but firms pay a fixed cost in each period that they operate a technology. They pay this period fixed cost in units of capital, the price of which is always in constant proportion to the price of the aggregate consumption basket; I normalize the price of capital to one. The period fixed cost for technology $v$ is given by:

$$f_{t+1,v} = \frac{Y_{t+1}}{\mu} v^\gamma.$$  \hfill (6)

The coefficient $Y_{t+1}/\mu$ simplifies the model: it causes period fixed costs to rise and fall with the state of the economy. Removing the assumption renders the technology choice problem (and therefore all uncertainty in the model) dynamic. Dynamic uncertainty is an attractive feature, but beyond the scope of this paper. In on-going work, I relax the simplifying assumption and study the dynamics of technology adoption, obsolescence, and endogenous uncertainty in an otherwise similar environment.

As it stands here, the technology choice problem is essentially static, and the rule for choosing technology sets that maximize expected profit is immediate: Operate any technology $v$ that satisfies:

$$\mathbb{E}_t \left[ m_{t,t+1}(\pi_{t+1,v}(\omega) - f_{t+1,v}) \right] > 0,$$ \hfill (7)

where $m_{t,t+1}$ is the household’s stochastic discount factor.

Finally, an aim of this paper is to characterize the stochastic properties of firm-level and economy-wide productivity aggregates, but technical challenges arise when each firm’s technology set is a continuous subset of $\mathcal{V}$. Aggregation then requires integrating over uncountable sets of random variables, and care must be taken to preserve randomness in the aggregates. To this end, I make the following assumption on technological productivity:

$$z_{t,v}^{\theta-1} := \epsilon_{t,v} \quad \forall v \in \mathcal{V},$$ \hfill (8)
with \( \{ \epsilon_{t,1}, \epsilon_{t,2}, \ldots \} \) a countable set of random variables. Think of the \( \epsilon_{t,n} \)'s as fundamental technologies upon which production technologies represented by the \( z_{t,v} \)'s are built. I make the following assumptions on the fundamental technologies: for all \( n, m \in \mathbb{N} \) and \( s, t \in \mathbb{Z} \), with \( n \neq m \) and \( s \neq t \), assume that \( \mathbb{E}[\epsilon_{t,n}] = \mu_{\epsilon} \), \( \text{Var}(\epsilon_{t,n}) = \sigma_{\epsilon}^2 \), and \( \text{Cov}(\epsilon_{t,n},\epsilon_{t,m}) = \text{Cov}(\epsilon_{s,n},\epsilon_{t,n}) = 0 \). This construction is a special case of the constructions considered in Al-Najjar (1995), and is specifically designed to preserve risk in continuum economies.\(^3\)

### 3.2 Households

A representative household buys baskets of consumption good varieties. To purchase these baskets, the household uses income derived from three sources: labor, physical capital, and financial capital. The household sells a fixed quantity of labor \( L \) to firms each period in return for a wage, it rents its current stock of physical capital \( K_t \) to firms in return for interest, and finally, it earns dividends and financial capital gains from equity shares in the firms that produce consumption goods. In addition to purchasing consumption goods, the household uses its income to maintain its equity portfolio and its stock of physical capital.

Physical capital is long-lived, subject to depreciation at rate \( \delta \), and supplied by a representative producer that takes the price of capital goods as given. The household’s stock of capital evolves according to

\[
K_{t+1} = I_t + (1 - \delta)K_t. \tag{9}
\]

A budget constraint summarizes the household’s sources and uses of funds, and binds in each period. Express prices and profits in units of the aggregate consumption basket and write the

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\(^3\)Al-Najjar considers collections of random variables \( f \) indexed by the measure space \( (T, \mathcal{T}, \tau) \), where \( T = [0, 1] \) is a continuous parameter space, and \( f_t = g_t + h_t \), with aggregate component \( g_t = \sum_{k=1}^{\infty} \beta_k \eta_k \), \( \{ \eta_1, \eta_2, \ldots \} \) a set of orthonormal random variables, and idiosyncratic component \( h_t \) such that \( \mathbb{E}[xh_t] = 0 \) \( \tau \)-a.e. for any random \( x \) defined on the same probability space as \( h_t \). In Al-Najjar’s notation, I consider the case where \( h_t = 0 \) \( \tau \)-a.e., and \( \beta_k := \beta_k(t) = 1 \) if \( k - 1 < t \leq k \) and zero otherwise. Here, \( \mathcal{V} \) corresponds to \( T, v \) to \( t \), \( z_{t,v} \) to \( g_t \), and \( \epsilon_{s,n} \) to \( \eta_k \). It’s worth noting that the special case I consider extends trivially using Al-Najjar’s construction to cases that feature idiosyncratic shocks to individual technologies or individual firms, and to cases where shocks are correlated across individual technologies.
constraint in period $t$ as

$$w_t L + r_t K_t + \int_\Omega [\Pi_t(\omega) - F_t(\omega)] S_t(\omega) \lambda(d\omega) = C_t + I_t + \int_\Omega V_t(\omega)[S_{t+1}(\omega) - S_t(\omega)] \lambda(d\omega),$$

(10)

where producer $\omega$ has market value $V_t(\omega)$ and pays dividends equal to its net profit $\Pi_t(\omega) - F_t(\omega)$ each period, $F_t(\omega) = \int_{V(\omega)} f_{t,v} \lambda(dv)$, and where $S_t(\omega)$ represents the household’s ownership share in firm $\omega$. The household maximizes expected discounted utility over its infinite lifetime by solving

$$\max_{C_s, K_{s+1}, S_{s+1}(\omega)} \left\{ \sum_{s=t}^{\infty} \beta^{s-t} u(C_s) \right\}$$

s.t. (10) and (9) $\forall s \geq t$,

(11)

where the utility function $u(C_s) = \ln(C_s)$.

The consumption basket $C_t$ represents an aggregate of all consumption good varieties available in period $t$. In this context, the aggregate should capture the household’s love of product diversity, which the many final goods producers and production technologies provide, as well as the household’s tendency to substitute away from expensive goods. I use the well-known Dixit and Stiglitz (1977) aggregator for this purpose, which has the advantage of tractability.

The consumption basket is given by

$$C_t = \left[ \left( \int_{\Gamma \forall v(\omega)} [c_{t,v}(\omega)]^{\frac{\theta-1}{\theta}} \lambda(dv,\omega) \right)^{\frac{\theta}{\theta-1}} \right],$$

(12)

where $c_{t,v}(\omega)$ denotes household demand for the variety that firm $\omega$ produces using technology $v$, and $\theta$ denotes the symmetric price elasticity of substitution between any two varieties. Write the
corresponding price index as

$$1 = \left[ \iiint_{\Omega \nu(\omega)} [p_{t,v}(\omega)]^{1-\theta} \lambda(dv d\omega) \right]^{1/\theta},$$

(13)

where $p_{t,v}(\omega)$ is the price that firm $\omega$ charges for the variety it produces with technology $v$ relative to the price of the consumption basket. Finally, write the household’s demand for each variety as

$$c_{t,v}(\omega) = [p_{t,v}(\omega)]^{-\theta} C_t.$$

(14)

### 3.3 Capital goods sector

For convenience, I model capital goods production separately from consumption goods production. The convenience of employing a two-sector model lies in reducing the number of state variables in the technology choice problem, and in simplifying aggregation. Separating the sectors serves no purpose beyond this convenience, so a basic specification for capital goods production will do.

A representative and privately-owned firm supplies the household with capital goods. As in the consumption goods sector, the firm here combines labor and capital in a constant-returns-to-scale production function with a stochastic productivity multiplier. The productivity multiplier here is an aggregate of the technological and firm-specific productivities of final goods producers. Because the consumption and capital goods sectors are subject to the same aggregate fluctuations, the price of the consumption basket correlates perfectly with that of the capital good. The relative price is therefore constant and equal to the markup in the consumption goods sector. I normalize the relative price to one by choice of units for the capital good.

The representative capital producer maximizes profit by choosing capital and labor inputs optimally, taking all prices as given. Write the firm’s production function as $\tilde{I}_t = Z_t(k_t)^{\alpha}(l_t)^{1-\alpha}$, gross profit function as $\pi_t = \tilde{I}_t - r_t k_t - w_t l_t$, and profit maximization problem as:

$$\max_{k_t, l_t} \pi_t,$$

(15)
where maximization is subject to the production and gross profit functions above, and where $Z_t$ is specified in more detail in Proposition 4.1. See appendix subsubsection A.1.1 for optimality conditions from decision problems (5), (11), and (15).

3.4 Equilibrium

An equilibrium consists of a set of real prices $\{P_{t,v}(\omega)\}_{\omega \in \Omega}$ for consumption goods, real factor market prices $w_t$ and $r_t$ for labor and capital, and real firm values $\{V_t(\omega)\}_{\omega \in \Omega}$ at which the household budget constraint is satisfied, the consumption and capital goods markets clear, capital and labor factor markets clear, the stock market clears, optimality conditions (35), (36), (38), (39), and (40) (found in appendix subsubsection A.1.1) are satisfied, and firm technology sets $\{V(\omega)\}_{\omega \in \Omega}$ include all technologies that satisfy the adoption rule (7) and none that don’t. See appendix subsubsection A.1.3 for details and steady-state equilibrium expressions.

4 Main Propositions

This section presents propositions communicating the key features of the model: first, how the technology choice problem works, second, how uncertainty is determined endogenously, and third, how covariance affects risk. The proofs are straightforward but tedious, and I provide them with some discussion in the appendix.

Proposition 4.1 characterizes aggregation. It states that the model can be aggregated in two ways: over technologies, and over firms. It states that special productivity averages, as in Melitz (2003), summarize completely the technological and firm-specific heterogeneity in the economy. Proposition 4.2 characterizes the firm’s problem of choosing a technology set. The proposition states that profit-maximizing firms use all available technologies below a firm-specific cost threshold, and that high-productivity firms have higher cost thresholds. Conveniently, the firm-specific cost threshold is enough to characterize the firm’s technology set completely. Proposition 4.3 highlights how firm technology choices endogenize uncertainty in the model. The proposition gives closed-form expressions for the endogenous first and second moments of firm and aggregate productivity.
distributions, assuming firms operate the technologies they would choose in non-stochastic steady state. As it happens, these technology sets are first-order approximations to the sets firms would choose in a stochastic world. Proposition 4.4 characterizes comovement, and highlights qualitative features the model shares with the data. I give closed-form expressions for endogenous firm-aggregate productivity covariance and firm-firm productivity correlation, as well as a second-order approximate expression for covariance-over-value. Where possible, I describe how the relevant statistics vary with firm-specific productivity, and how they respond to structural changes in the economy. Finally, Proposition 4.5 shows how covariance risk affects stock returns. It gives a second-order approximate expression for expected excess returns in terms of firm-aggregate productivity covariance over market value, and states that expected returns are lower for high-productivity firms.

4.1 Aggregation

One aim of this model is to characterize endogenous uncertainty at the firm-level and for the economy. Because each firm chooses a different sets of technologies, and each technology is a distinct source of randomness, aggregation over technologies and firms is a convenient way to summarize technological uncertainty. Fortunately, heterogeneity takes a limited form in the model, so aggregation is straightforward, both over technologies and over firms. Thus, production can be viewed in three ways: as individual varieties produced by individual technologies, as baskets of goods produced by individual firms, or as an economy-wide consumption basket. This makes the model environment ideal for studying microeconomic sources of aggregate fluctuations. Proposition 4.1 characterizes aggregates in terms of special productivity variables. The aggregation strategy I employ was first developed by Houthakker (1955), and later used to good effect by Melitz (2003) to study trade, and Ghironi (2006) to study macroeconomic dynamics and trade. Here, the aggregation occurs in two stages.

Proposition 4.1 (Aggregation). A productivity aggregate over technologies summarizes all of the
technological heterogeneity within an individual firm $\omega$:

$$Z_t(\omega) = \left[ \int_{\mathcal{V}(\omega)} [z(\omega)z_{t,v}]^{\theta-1} \lambda(dv) \right]^{\frac{1}{\theta-1}}. \quad (16)$$

A productivity aggregate over firms summarizes all of the firm-specific and technological heterogeneity within the consumption goods sector:

$$Z_t = \left[ \int_{\Omega} Z_t(\omega)^{\theta-1} \lambda(d\omega) \right]^{\frac{1}{\theta-1}}. \quad (17)$$

Aggregate factor demands, production, and profit can be written in terms of aggregate productivities and variables that do not vary across entities. The proof is in the appendix.

The aggregate expressions for factor demands, output, and profit take simple forms, and suggest a close relationship between the economy here with multi-product, multi-technology firms, and simpler production economies with single-technology, single-product firms. For instance, the basic Cobb-Douglas production structure is preserved in aggregation. The heterogeneous technologies do little more than constrain the stochastic process driving aggregate productivity, as compared with the simpler economies; and the constraints allow the model to match cross-sectional features of firm-aggregate covariance, first and second moments of productivity, and to explain systemic risk and aggregate fluctuations.

4.2 Technology Choice

Firms choose their technology sets in the model, and because technology is the only source of randomness, the choices firms make ultimately determine the probability distributions of random productivity shocks at the firm level and for the aggregate economy. The technology choice problem is static under the simplifying assumptions made on the primitives of the model. Firms operate any technology that they expect will earn them positive net profit in the following period, after paying fixed operating costs in units of physical capital. The fixed cost differs across technologies,
low for some, high for others, and most firms operate only a subset of the available technologies in \( V \). Because some firms have higher firm-specific productivity, they are able to operate profitably at higher fixed costs, and therefore choose to operate a greater number of technologies. Proposition 4.2 characterizes the technology choices that individual firms make.

**Proposition 4.2 (Technology sets).** In a non-stochastic steady state, any firm \( \omega \) with productivity \( z(\omega) \geq \overline{z} \) chooses technology set \( V(\omega) = \{ v \in V : \underline{v} \leq v \leq \overline{v}(\omega) \} \), where the endogenous cut-offs \( \overline{z} \) and \( \overline{v}(\omega) \) are given by:

\[
\overline{z} = \left( \frac{\theta}{\mu_c} \right)^{\frac{1}{\gamma}} \tag{18}
\]

\[
\overline{v}(\omega) = \left( \frac{\mu_c}{\theta} \right)^{\frac{1}{\gamma}} z(\omega)^{\frac{\gamma-1}{\gamma}}. \tag{19}
\]

The above cut-offs are also first-order approximate to those that obtain in a stochastic world. Firms with \( z(\omega) < \overline{z} \) do not produce. Under parameter restrictions, firms \( \omega_1 \) and \( \omega_2 \) with productivities \( \overline{z} < z(\omega_1) < z(\omega_2) \) choose technology sets such that \( V_t(\omega_1) \subset V_t(\omega_2) \). The proof is in the appendix.

Panel (a) of Figure 2 illustrates the technology choice problem of an arbitrary firm \( \omega \). Available technologies are arranged along the horizontal axis, and expected gross profit and operating cost

---

**Figure 2 – Geometry of Technology Sets** The figure on the left shows how firm \( \omega \)'s technology set is determined by the intersection of the gross profit and period fixed-cost curves. The firm can profitably produce with technologies to the left of the intersection. The figure on the right shows how technology sets overlap for firms with different productivities. The low-productivity firm \( \omega_1 \)'s technology set is a proper subset of the high-productivity firm \( \omega_2 \)'s technology set.
measured by the vertical axis. The assumptions on the fundamental productivity shocks $\epsilon_{s,[v]}$ imply that the expected gross profit curve is horizontal to a first-order approximation, while the operating cost curve is assumed to increase in the technology index $v$. The intersection of gross profit and operating cost curves marks firm $\omega$’s cost threshold, and the firm cannot profitably diversify into business lines above this cost threshold.

### 4.3 Firm and Aggregate Productivity

The expected values and variances of the random firm-level and economy-wide productivity aggregates are endogenous, because they depend on the technology sets that firms choose to operate, and firms make this choice endogenously. I use the cost threshold from the technology choice problem in Proposition 4.2 to derive explicit expressions for the expected values and variances of firm-level and economy-wide productivity.

**Proposition 4.3** (Productivity First and Second Moments). Let technology sets be those that firms choose in the non-stochastic steady state. Then the first and second moments of firm-level productivity, denoted $\mu(\omega) = \mathbb{E}_t [Z_t(\omega)^{\theta-1}[t+1]]$ and $\sigma^2(\omega) = \operatorname{Var}_t (Z_t(\omega)^{\theta-1}[t+1])$, are given by:

\[
\mu(\omega) = \mu_\epsilon z(\omega)^{\theta-1} \left[ \left( \frac{z(\omega)}{\tilde{z}} \right)^{\zeta_{\omega}} - 1 \right]
\]

\[
\sigma^2(\omega) = \sigma_\epsilon^2 z(\omega)^{2(\theta-1)} \left[ \left( \frac{z(\omega)}{\tilde{z}} \right)^{\zeta_{\omega}} - 1 \right].
\]

The first and second moments of aggregate productivity, denoted $\mu = \mathbb{E} [Z_t^{\theta-1}]$ and $\sigma^2 = \operatorname{Var} (Z_t^{\theta-1})$, respectively, are given by:

\[
\mu = \mu_\epsilon \tilde{z}^{\theta-1}
\]

\[
\sigma^2 = \sigma_\epsilon^2 \tilde{z}^{2[\kappa-(\theta-1)]}.
\]

Under parameter restrictions, the first and second moments of all productivity aggregates are
positive and finite, and for any firms $\omega_1$ and $\omega_2$ with $z(\omega_1) < z(\omega_2)$, it holds that $\mu(\omega_1) < \mu(\omega_2)$ and $\sigma^2(\omega_1) < \sigma^2(\omega_2)$. The proof is in the appendix.

4.4 Productivity Comovement

Covariance in the model arises from overlapping technology sets: when firms use some of the same technologies, they are subject to some of the same random fluctuations in technological productivity, and their activities covary. Panel (b) of Figure 2 illustrates this effect. Because high-productivity firms have larger technology sets and produce at greater scale, they have more overlap with other firms, and higher covariances.

High-productivity firms are also more profitable than other firms, and so have higher market values, but above a low threshold, the ratio of covariance to market value falls in firm productivity. The ratio is falling because more productive firms use some technologies that few other firms use; these technologies generate profit for the firm, which raises market value, but contribute little to the firm’s covariance with aggregate productivity. As a firm’s productivity approaches the productivity cut-off $z$ from above, both its covariance and its market value race to zero.
Proposition 4.4 (Firm-Aggregate Covariance). Let technology sets be those that firms choose in the non-stochastic steady state. Then the covariance between firm and aggregate productivity, denoted by $\sigma_{\omega,\Omega}(\omega) = \text{Cov} \left( Z_t(\omega)^{\theta-1}, Z_t^{\theta-1} \right)$, is given by

$$
\sigma_{\omega,\Omega}(\omega) = z(\omega)^{\theta-1} \zeta_{\omega,\Omega} \left[ 1 - \left( \frac{\bar{z}}{z(\omega)} \right)^{\zeta_{\omega,\Omega}^2} \right].
$$

(24)

The covariance between firm and aggregate productivity, expressed as a fraction of firm market value, is given by

$$
\frac{\sigma_{\omega,\Omega}(\omega)}{V_t(\omega)} = \frac{\zeta_{\omega,\Omega} \left[ 1 - \left( \frac{\bar{z}}{z(\omega)} \right)^{\zeta_{\omega,\Omega}^2} \right]}{Y_t \left( \zeta_{V1} \left( \frac{\bar{z}(\omega)}{\bar{z}} \right) \zeta_{V2} + \zeta_{V3} \left( \frac{1}{z(\omega)} \right) \zeta_{V4} - \left( \frac{1}{\bar{z}} \right)^{\theta-1} \right)}.
$$

(25)

Under parameter restrictions, covariance-over-value falls for all $z(\omega)$ above a threshold. The ratio falls in the level of aggregate output. The proof is in the appendix.

4.5 Stock returns

Covariance-over-value measures firm-level systemic risk in the model. In theory, a risk-averse investor should be willing to accept lower returns from high-productivity firms, because the activities of these firms covary less with aggregate activity, relative to the discounted future profit investors expect. As in the classical capital asset pricing models, and the consumption based models, I am able to directly express expected stock returns in terms of covariance—in this case, firm-aggregate covariance over market value.

Proposition 4.5 (Stock returns). Let technology sets be those that firms choose in the non-stochastic steady state. Then firm $\omega$’s expected excess return is approximated to a second order by

$$
E \left[ r_t(\omega) - r_{f,t} \right] \approx \zeta_{r1} \frac{\mu(\omega)}{V_t(\omega)} + \zeta_{r2} \frac{\sigma_{\omega,\Omega}(\omega)}{V_t(\omega)}.
$$

(26)

where I define firm $\omega$’s return as $r_t(\omega) = [V_{t+1}(\omega) + \Pi_{t+1}(\omega) - F_{t+1}(\omega)] / V_t(\omega)$, and the risk-free rate
as $r_{f,t} = m_{t,t+1}^{-1}$. Under parameter restrictions, expected excess returns decrease in firm productivity $z(\omega)$ for all $z(\omega)$ above a threshold. The proof is in the appendix.

The above propositions explain the model’s mechanism, and highlight key results. In particular, the propositions illustrate how the technology choice mechanism leads to endogenous first and second moments of firm-level and aggregate productivity, and endogenous covariance between firm and aggregate values. The propositions also show how covariance affects systematic risk and stock returns.

5 Empirical Work

The theoretical model builds on motivating evidence from Compustat presented in Figure 1. In this section, I describe the data and rolling-window covariance calculations used to produce the figure, provide detailed results with standard errors in Table ??, and discuss alternative empirical approaches. I describe my estimation of total factor productivity, which is based on Olley and Pakes (1996) and İmrohoroğlu and Tüzel (2014). Finally, I test whether the documented covariance patterns are explained by business-line diversification, after controlling for financial strength, potential aggregate or industry sources of comovement, and firm-specific characteristics.

5.1 Data Description

For accounting data I use the WRDS Compustat North America Fundamentals Annual database, which, after cleaning, covers about 67,693 observations on 6,462 firms over the period 1966–2015. Figure 4 illustrates some features of the sample. The Compustat data cover foreign and domestic firms that are or were publicly traded in the United States on NYSE, NASDAQ, or AMEX. The key accounting variables I use for productivity estimation and the aggregate variance decompositions are employment ($\text{EMP}$); net property, plant and equipment ($\text{PPENT}$) as a measure of physical capital; depreciation ($\text{DP}$) and accumulated depreciation ($\text{DPACT}$) to estimate the age of the capital stock; net sales ($\text{SALE}$), which I refer to simply as sales; operating income before before depreciation ($\text{OIBDP}$), which I refer to as profit; and fiscal year closing share price ($\text{PRCC}_F$) and common shares outstanding
(CSHD) to compute market value. I use the following additional variables to construct controls for financial strength in regressions: total debt (DT) and common equity (CEQ) to compute leverage; interest expense (INT) to compute the interest coverage ratio; and cash and short term investments (CHE), receivables (RECT), and current liabilities (LCT) to compute the quick ratio. I largely follow İmrohoroğlu and Tüzel (2014) and Covas and den Haan (2011) in cleaning the Compustat data. I drop financial and utilities firms, observations prior to 1961, and observations with missing values on any of the variables used in productivity estimation or rolling-window covariances. I also drop firms involved in large mergers, and the smallest 10% of firms by market value. I use data on nominal GDP, and GDP and non-residential investment deflators from the Bureau of Economic Analysis, and Social Security Administration data on national average wage.

5.2 Productivity Estimation

A broad empirical literature on industry dynamics documents large differences in total factor productivity across firms. To name several contributions in this literature, Baily et al. (1992), Bartelsman and Doms (2000), and Foster et al. (2001) provide evidence for U.S. manufacturing establishments, Olley and Pakes (1996) look at plant-level productivity in telecommunications and develop their now widely-used productivity estimation procedure, and Bartelsman, Haltiwanger, et al. (2009) provide cross-country evidence. Productivity also plays an important role in theory: firm-level productivity shocks are the stochastic driver of an important class of models used to study industry dynamics, beginning with Hopenhayn (1992). For consistency with this established literature, and for consistency with my theoretical model, I also characterize the firm cross-section by estimated total factor productivity, rather than sales, employment, or market value.

I estimate firm-level productivity as the unexplained firm-level residual from panel regressions of value added on employment (EMP) and capital (PPENT), where value added is constructed from sales (SALE), profit (OIBDP), and employment (EMP), along with Social Security Administration data on average wage. I follow İmrohoroğlu and Tüzel (2014) closely in my productivity estimation procedure, which consists of estimating a Cobb-Douglas production function in log form, after controlling for simultaneity and selection bias that arise from the forecastability of future firm-level
Table 1 – Summary Statistics for Compustat Firms. Firms are grouped into productivity deciles, each decile forming a column. All statistics are averaged or aggregated within decile, then averaged over time. Averages are reported over the forty-year period 1976–2015, and the two twenty-year periods 1976–1995 and 1996–2015. Compustat segment data is unavailable prior to 1976. The first row of each panel shows decile productivity means relative to the average firm. The next row shows the average number of segments reported by firms. The last three rows show aggregate decile shares of total employment, total sales, and total profit and sum to one hundred plus rounding errors.

<table>
<thead>
<tr>
<th>1976–2015</th>
<th>Low</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>High</th>
</tr>
</thead>
<tbody>
<tr>
<td>Productivity</td>
<td>0.39</td>
<td>0.50</td>
<td>0.58</td>
<td>0.66</td>
<td>0.74</td>
<td>0.84</td>
<td>0.96</td>
<td>1.13</td>
<td>1.42</td>
<td>2.78</td>
</tr>
<tr>
<td>— variance</td>
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<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.02</td>
<td>0.16</td>
<td>0.06</td>
<td>0.51</td>
<td>1.21</td>
<td>8.00</td>
</tr>
<tr>
<td>— covariance</td>
<td>0.01</td>
<td>0.06</td>
<td>0.06</td>
<td>0.11</td>
<td>0.14</td>
<td>0.25</td>
<td>0.48</td>
<td>0.84</td>
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<tr>
<td>— cov-over-val</td>
<td>2.46</td>
<td>2.54</td>
<td>1.09</td>
<td>1.26</td>
<td>0.75</td>
<td>0.65</td>
<td>0.53</td>
<td>0.30</td>
<td>0.08</td>
<td>0.34</td>
</tr>
<tr>
<td>Number of segments</td>
<td>1.53</td>
<td>1.53</td>
<td>1.62</td>
<td>1.69</td>
<td>1.75</td>
<td>1.88</td>
<td>2.04</td>
<td>2.21</td>
<td>2.49</td>
<td>2.55</td>
</tr>
<tr>
<td>Employment share</td>
<td>0.39</td>
<td>0.83</td>
<td>1.42</td>
<td>2.19</td>
<td>3.50</td>
<td>5.42</td>
<td>8.82</td>
<td>14.65</td>
<td>25.44</td>
<td>37.34</td>
</tr>
<tr>
<td>Sales share</td>
<td>0.26</td>
<td>0.55</td>
<td>0.92</td>
<td>1.46</td>
<td>2.23</td>
<td>3.51</td>
<td>6.21</td>
<td>11.27</td>
<td>21.29</td>
<td>52.29</td>
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<tr>
<td>Profit share</td>
<td>-0.02</td>
<td>0.10</td>
<td>0.27</td>
<td>0.56</td>
<td>1.05</td>
<td>1.90</td>
<td>3.58</td>
<td>7.47</td>
<td>18.33</td>
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<table>
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<tr>
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</tr>
<tr>
<td>— variance</td>
</tr>
<tr>
<td>— covariance</td>
</tr>
<tr>
<td>— cov-over-val</td>
</tr>
<tr>
<td>Number of segments</td>
</tr>
<tr>
<td>Employment share</td>
</tr>
<tr>
<td>Sales share</td>
</tr>
<tr>
<td>Profit share</td>
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<table>
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</tr>
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<td>— variance</td>
</tr>
<tr>
<td>— covariance</td>
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<tr>
<td>— cov-over-val</td>
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<tr>
<td>Number of segments</td>
</tr>
<tr>
<td>Employment share</td>
</tr>
<tr>
<td>Sales share</td>
</tr>
<tr>
<td>Profit share</td>
</tr>
</tbody>
</table>
productivity, and from entry and exit of firms in the Compustat sample. The estimation model is given by:

\[ y_{i,t} = \beta_l l_{i,t} + \phi(i_{i,t}, k_{i,t}) + \eta_t \]  

(27)

\[ P_{i,t} = \mathcal{P}(i_{i,t}, k_{i,t}) \]  

(28)

\[ y_{i,t+1} - \beta_l l_{i,t+1} = \beta_k k_{i,t+1} + g(P_{i,t}, \phi_{i,t} - \beta_k k_{i,t}) + \xi_{i,t+1} + \eta_{i,t+1}. \]  

(29)

The estimation proceeds in stages. In a first stage, (27) is estimated by least squares, where \( \phi(i_{i,t}, k_{i,t}) \) controls for the forecastable component of firm productivity, and is approximated by a polynomial in \( i_t \) and \( k_t \), denoting a firm’s current investment and lagged capital stock. I include time-industry controls to prevent aggregate shocks from influencing estimates in the first stage.

In the second stage, each firm’s probability of exit is estimated by equation (28) using probit, where \( \mathcal{P}(i_{i,t}, k_{i,t}) \) is approximated by a polynomial in \( i_t \) and \( k_t \). Finally, (29) is estimated by non-linear least squares, using estimates from stages one and two for \( P_{i,t} \) and \( \phi_{i,t} \), and approximating \( g(P_{i,t}, \phi_{i,t} - \beta_k k_{i,t}) \) non-parametrically. The non-parametric functions \( \phi \), \( \mathcal{P} \), and \( g \) derive from the reduced-form theoretical model in Olley and Pakes (1996).

İmrohoroğlu and Tüzel (2014) use an expanding estimation window to prevent information that would have been unavailable to market participants in a particular period from distorting results when they combine estimated productivity with financial market data. I find that the expanding window approach leads to large differences in the volatility of production function estimates in earlier periods relative to later periods. This increased volatility biases the rolling-window covariance estimates in early years, so I instead use the full sample period to estimate production function parameters, and then compute productivity as the residual each period.

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4I am grateful to Selale Tüzel for making her productivity estimation code available online, key parts of which I have used in this project.
Figure 4  -  Descriptive Statistics for the Compustat Sample, 1966–2015. The scatter plot on the left relates log firm size to log total factor productivity for the year 2015. Size is measured by net sales, in millions of 2009 dollars, and total factor productivity is estimated by the Olley and Pakes (1996) method. The black dots, from left to right, are Starbucks and Boeing. I apply the logarithmic transformation because both size and productivity distributions are highly skewed to the right. The middle figure plots the number of firms in the Compustat sample per year, and shows this number increasing rapidly in the early part of the sample. The rapid rise is partly due to the addition of NASDAQ in 1973, as Fama and French (1992) report. The figure on the right plots aggregate real value added for the Compustat sample as a fraction of U.S. real GDP. This fraction tends to rise with the number of firms in the sample. I drop financial and utilities firms, observations prior to 1961, observations with missing values on any of the variables used in productivity estimation or rolling-window covariances. I also drop firms in large mergers, and the smallest 10% of firms by market value.

5.3 Aggregate Variance Decomposition

The motivating evidence for this paper derives from the aggregate variance decomposition in equation (1). In this section I apply that to the sample of Compustat firms, using total factor productivity estimates to characterize the cross-section of firms. The decomposition answers three main questions: First, how much of the variance of aggregate growth rates in productivity, sales, and profit comes from firm-level covariances? Second, which firms in the productivity cross-section generate the most aggregate variance? Third, which firms generate the most aggregate variance relative to their market values? I find that pairwise covariance in firm-level variables accounts for most of the variance in aggregate growth rates, and that firms in more productive deciles tend to have higher firm-aggregate covariances. The relationship holds for productivity, sales, and profits growth. The relationship between productivity and covariance reverses when covariance is expressed per dollar of market value. I interpret this evidence as follows: high-productivity firms contribute more to aggregate variance, but carry less risk, dollar for dollar, than their less-productive competitors. Lastly, I find that firms of similar productivity tend to covary more in their economic activities than firms with
dissimilar productivity.

I compute sample covariances in rolling windows for all firm pairs in my Compustat sample, and all available years, then average or aggregate these as appropriate. My focus on covariance as a measure of comovement is motivated by equation (1), which expresses aggregate variance in terms of firm-level covariances, and by equation (2), which expresses expected returns in terms of a firm-level covariance per dollar of market value. The rolling-window approach has been employed in the economics literature by Comin and Mulani (2004) and Forbes (2012), and has advantages beyond its simplicity. First, it adds a time dimension to the covariances that would be absent if they were computed over the full sample period, and second, it limits the practical problem of characterizing differences in covariance across high- and low-productivity firm groups when firms frequently move between groups—a problem known as reclassification bias, and discussed in Moscarini and Postel-Vinay (2009).

I report covariance statistics for observations sorted into decades and for firms sorted into productivity decile in Table 2. Pooling by decile has been a common way to characterize the cross section in the finance literature since Fama and French (1992), and has been employed recently in Fama and French (2008), İmrohoroğlu and Tüzel (2014), and Fama and French (2016). In Table 3a the reported statistics are averages over all firms for each decade covered by the sample. In Table 3b the statistics are relative to the average firm each year, averaged over firms within productivity deciles, and averaged over years. Table 3a shows that over 80% of variance in aggregate productivity, sales, and profit growth is explained by firm-level covariances on average for every decade since 1975. Table 3b shows, for instance, that high-productivity firms contribute over six times what the average firm contributes to aggregate variance, but only around one-third as much per dollar of market value. The decade portfolios range from under 300 to over 1500 distinct firms, and the decile portfolios from 700 to 2,000 distinct firms.

To compute the sample variances and covariances, I use backward-looking rolling windows, and choose a window length close to the average length of the post-war U.S. business cycle, measured peak to peak. The average length is 68.5 months using National Bureau of Economic Research
Table 2 – Variance and Covariance Decomposition, 1966 to 2015. Compustat Annual Fundamentals North America. The table characterizes firm-level pairwise covariances over time and in the cross section of firms for productivity, sales, and profit growth. Panel (a) shows the average fraction of aggregate variance attributable to pairwise covariance during the decades ending 1975, 1985, 1995, 2005, and 2015. The last three rows of each panel summarize the firm sample. The rows show aggregate value added for the sample, expressed as a fraction of U.S. GDP, as well as firm and observation counts.

(a) Sum of Pairwise Covariances, Relative to Aggregate Variance

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Productivity</td>
<td>0.81</td>
<td>0.85</td>
<td>0.90</td>
<td>0.84</td>
<td>0.85</td>
</tr>
<tr>
<td>s.e.</td>
<td>(0.03)</td>
<td>(0.03)</td>
<td>(0.01)</td>
<td>(0.03)</td>
<td>(0.04)</td>
</tr>
<tr>
<td>Sales</td>
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<td>0.94</td>
<td>0.89</td>
<td>0.85</td>
<td>0.94</td>
</tr>
<tr>
<td>s.e.</td>
<td>(0.04)</td>
<td>(0.01)</td>
<td>(0.02)</td>
<td>(0.03)</td>
<td>(0.02)</td>
</tr>
<tr>
<td>Profit</td>
<td>0.69</td>
<td>0.85</td>
<td>0.88</td>
<td>0.83</td>
<td>0.72</td>
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<td>s.e.</td>
<td>(0.05)</td>
<td>(0.01)</td>
<td>(0.01)</td>
<td>(0.05)</td>
<td>(0.17)</td>
</tr>
</tbody>
</table>

Decade Descriptions

| Avg Fraction of GDP | 0.15 | 0.18 | 0.14 | 0.17 | 0.25 |
| Avg Firms per Year  | 279  | 1026 | 1357 | 1519 | 1655 |
| Firm-Year Observations | 5481 | 13452 | 13291 | 15261 | 16951 |

(b) Firm-Aggregate Covariance, Firm-Aggregate Covariance Over Market Value, Relative to Average Firm

<table>
<thead>
<tr>
<th>Covariance</th>
<th>Low</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>High</th>
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</thead>
<tbody>
<tr>
<td>Productivity</td>
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<td>0.06</td>
<td>0.06</td>
<td>0.11</td>
<td>0.14</td>
<td>0.26</td>
<td>0.46</td>
<td>0.84</td>
<td>1.63</td>
<td>6.43</td>
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<tr>
<td>s.e.</td>
<td>(0.01)</td>
<td>(0.00)</td>
<td>(0.01)</td>
<td>(0.01)</td>
<td>(0.01)</td>
<td>(0.05)</td>
<td>(0.02)</td>
<td>(0.10)</td>
<td>(0.14)</td>
<td>(0.27)</td>
</tr>
<tr>
<td>Sales</td>
<td>0.03</td>
<td>0.05</td>
<td>0.09</td>
<td>0.13</td>
<td>0.21</td>
<td>0.28</td>
<td>0.52</td>
<td>0.91</td>
<td>1.70</td>
<td>6.12</td>
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<tr>
<td>s.e.</td>
<td>(0.00)</td>
<td>(0.00)</td>
<td>(0.01)</td>
<td>(0.01)</td>
<td>(0.01)</td>
<td>(0.01)</td>
<td>(0.02)</td>
<td>(0.04)</td>
<td>(0.09)</td>
<td>(0.22)</td>
</tr>
<tr>
<td>Profit</td>
<td>0.01</td>
<td>0.02</td>
<td>0.04</td>
<td>0.06</td>
<td>0.11</td>
<td>0.18</td>
<td>0.45</td>
<td>0.85</td>
<td>1.46</td>
<td>6.85</td>
</tr>
<tr>
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<td>(0.00)</td>
<td>(0.01)</td>
<td>(0.01)</td>
<td>(0.01)</td>
<td>(0.02)</td>
<td>(0.04)</td>
<td>(0.05)</td>
<td>(0.09)</td>
<td>(0.29)</td>
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Covariance-Over-Value

<table>
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<th>Covariance</th>
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<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>High</th>
</tr>
</thead>
<tbody>
<tr>
<td>Productivity</td>
<td>1.97</td>
<td>2.53</td>
<td>1.07</td>
<td>1.22</td>
<td>0.77</td>
<td>0.67</td>
<td>0.51</td>
<td>0.32</td>
<td>0.09</td>
<td>0.35</td>
</tr>
<tr>
<td>s.e.</td>
<td>(1.29)</td>
<td>(0.35)</td>
<td>(0.19)</td>
<td>(0.11)</td>
<td>(0.10)</td>
<td>(0.08)</td>
<td>(0.11)</td>
<td>(0.19)</td>
<td>(0.03)</td>
<td>(0.06)</td>
</tr>
<tr>
<td>Sales</td>
<td>3.14</td>
<td>1.62</td>
<td>1.19</td>
<td>0.93</td>
<td>0.67</td>
<td>0.53</td>
<td>0.34</td>
<td>0.22</td>
<td>0.13</td>
<td>0.28</td>
</tr>
<tr>
<td>s.e.</td>
<td>(0.69)</td>
<td>(0.35)</td>
<td>(0.21)</td>
<td>(0.12)</td>
<td>(0.13)</td>
<td>(0.09)</td>
<td>(0.04)</td>
<td>(0.05)</td>
<td>(0.06)</td>
<td>(0.05)</td>
</tr>
<tr>
<td>Profit</td>
<td>1.43</td>
<td>3.64</td>
<td>1.73</td>
<td>0.81</td>
<td>0.79</td>
<td>0.61</td>
<td>0.34</td>
<td>0.27</td>
<td>0.10</td>
<td>0.28</td>
</tr>
<tr>
<td>s.e.</td>
<td>(1.65)</td>
<td>(0.47)</td>
<td>(0.21)</td>
<td>(0.16)</td>
<td>(0.12)</td>
<td>(0.10)</td>
<td>(0.06)</td>
<td>(0.04)</td>
<td>(0.04)</td>
<td>(0.06)</td>
</tr>
</tbody>
</table>

Decile Descriptions

| Avg Fraction of GDP | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Avg Firms per Year  | 1644 | 1908 | 1903 | 1863 | 1769 | 1587 | 1400 | 1159 | 891  | 701  |
| Firm-Year Observations | 6465 | 6442 | 6444 | 6444 | 6435 | 6451 | 6448 | 6440 | 6446 | 6421 |
dates, and I round up to six years because the Compustat data is annual. Longer windows give more stable sample covariances, as you would expect, but they also reduce the number of firms in the sample, because firms with too few within-window observations must be excluded. In practice, varying the window length between five and ten years makes little difference to the main conclusions because the panel width is large. I choose a backward-looking window, rather than a centered one, to avoid introducing future information that would be unavailable to investors in the present. This fact explains why characteristic spikes appear to lag familiar dates of economic crises in the time series plots in Figure 1.

I define the rolling-window sample variance and covariance for arbitrary variable firm variable $x_\omega$ in the usual way. For rolling window $W_t = \{t - w, \ldots, t - 1, t\}$, $s \in W_t$, and $\omega_1, \omega_2$ firm indices:

$$\text{Var}_t (x_\omega) = \frac{1}{w} \sum_{s \in W_t} (x_{\omega,s} - \bar{x}_{\omega,s})^2,$$

$$\text{Cov}_t (x_{\omega_1}, x_{\omega_2}) = \frac{1}{w} \sum_{s \in W_t} (x_{\omega_1,s} - \bar{x}_{\omega_1,s})(x_{\omega_2,s} - \bar{x}_{\omega_2,s}).$$  

Entry, exit, and missing data are common in Compustat. To deal with this problem, I exclude firms with data gaps from the rolling-window calculations. I do this for consistency with Comin and Mulani (2004), and because my model doesn’t feature entry and exit. Using only the subsample of firms with non-missing values over window $W_t$, and denoting this set of firms $\Omega_{W_t}^n$ ($n$ for non-missing), the variance of aggregate variable $X = \sum_{\Omega_{W_t}^n} x_\omega$ becomes

$$\text{Var}_t (X) = \text{Var}_t \left( \sum_{\Omega_{W_t}^n} x_\omega \right)$$

$$= \sum_{\Omega_{W_t}^n} \text{Var}_t (x_\omega) + \sum_{\Omega_{W_t}^n} \sum_{\Omega_{W_t}^n \setminus \{\omega\}} \text{Cov}_t (x_\omega, x_{\omega'}).$$

Requiring consecutive years of non-missing observations is a costly convenience: it omits an important source of aggregate variance, and it biases the sample of firms. Entry and exit as a source of variance has recently been emphasized by Ghironi and Melitz (2005), Bilbiie et al.

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5The NBER dates were here http://www.nber.org/cycles.html as of August 2018.
(2012), Carvalho and Grassi (2015), and Clementi and Palazzo (2016). To see the implications of omitting entry and exit for this study, consider the variance of aggregate variable \( X' = \sum_{i} \omega_{i} x_{i} \), where \( \Omega_{i} \) is the set of firms with at least one observation in window \( W_{i} \). Now, \( \text{Var}_{t}(X') = \text{Var}_{t}(X) + \text{Var}_{t}\left( \sum_{i} \Omega_{i} \omega_{i} x_{i} \right) + 2 \text{Cov}_{t}(X, \sum_{i} \Omega_{i} \omega_{i} x_{i}) \), where \( \Omega_{i} \) is the set of firms with missing values, and where \( \Omega_{i}^{m} \) and \( \Omega_{i}^{m} \) constitute a partition of \( \Omega_{i} \). Under the rolling-window covariance procedure I adopt here, the second two terms of this expression are ignored. Entry and exit are beyond the scope of this paper, but I consider them in a companion paper that models a dynamic technology choice problem in a similar environment. Table 2 summarizes the results of the decomposition for firms grouped by decade and productivity decile, for productivity, sales, and profit growth.

5.4 Regressions

The regressions in this section test key predictions of the model. The mechanism in the model is business-line diversification, where a business line consists of a technology and the consumption good it produces. Covariance arises when firms have overlapping technology sets, and for high-productivity firms with many business lines, the overlap is larger. But because high-productivity firms use some technologies that few other firms use, these technologies add more to the value than to firm-aggregate covariance of high-productivity firms. The model makes two predictions based on this logic:

Hypothesis 1: *Ceteris paribus, covariance between firm and aggregate growth rates increases in a firm’s level of diversification.*

Hypothesis 2: *Ceteris paribus, covariance per unit of market value between firm and aggregate growth rates decreases in a firm’s level of diversification.*

The question is whether the business-line diversification theory can account for a non-negligible amount of the cross-sectional variation in covariances and covariances-over-value between firm and aggregate growth rates in the Compustat data, after controlling for other sources of firm-aggregate covariance. To test the hypotheses, I run regressions on firms’ reported number of Compustat business segments as a coarse measure of diversification, while controlling for alternative explanations:
the financial strength of firms, aggregate factors like money supply, interest rates, or the price oil, and firm-specific effects like industry, age and size. I find tentative support for the model’s two predictions. Results are reported in Table 3, and the regression equation is given below:

$$\frac{\text{Cov}_t(x_\omega, X)}{\text{Base}} = \beta_0 + \beta_1 (\text{Diversification}_\omega) + \beta_2 (\text{Financial Strength}_\omega) + \beta_3 (\text{Aggregate Shocks}) + \beta_4 (\text{Industry Controls}) + \beta_5 (\text{Other Controls}) + \epsilon_{t, \omega},$$

where Base is either aggregate variance or firm value.

The equation constitutes an estimated dependent variable model, because the left-hand side covariances are rolling-window estimates. Hausman (2001) and Lewis and Linzer (2005) discuss the econometric issues that arise in estimated dependent variable models like these. Hausman reminds us that an estimated dependent variable doesn’t bias results as long as the classical ordinary least squares assumptions are met. Of course, they may not be met: At issue is whether the firm covariances have sampling errors that vary in the cross-section of firms. Lewis and Linzer find that feasible generalized least squares works well if heteroskedasticity in the standard errors of the estimated dependent variable are large; otherwise, they recommend ordinary least squares using White’s estimator (1980) for consistency. I take the latter approach here.

I review some alternative explanations for covariance that are common in the literature, before turning to a discussion of results from the main specification of the regression model.

Diversification. The main paper here is Villalonga (2004), who finds that diversified firms earn a value premium. Wang (2012) argues that firms with multiple business segments have access to a broader set of growth options; if growth options differ in their risk properties, then holding a broad set of growth options may lead to diversification benefits. Note that my framework can be interpreted as endogenizing the risk properties of different business segments, which correspond to the growth options in Wang (2012). Empirically, I use the number of business lines firms report in Compustat segment data to proxy for product and technology diversification.

Financial Strength. Firms that rely on funds from banks and financial markets to run their
businesses may respond similarly when the cost or availability of funds changes. Fama and French (1992) and Gertler and Gilchrist (1994) have argued that changes in access to external funds most affect small and financially weak firms, but recent work by Chari et al. (2007) and Crouzet and Mehrotra (2017) calls this view into question. I use firm leverage and liquidity ratios to control for financial strength. For leverage, I follow Rajan and Zingales (1995) in defining three different ratios, the primary one being debt-to-equity. I follow Davydenko (2012) for my main measure of liquidity: cash and accounts receivable over current liabilities, and use the interest coverage ratio as an alternative measure. Crouzet and Mehrotra (2017) use cash to assets.

**Aggregate Shocks.** Cochrane (1994) offers a wide-ranging discussion and empirical evaluation of the most common aggregate shocks that macroeconomists rely on. In addition to technology and consumption shocks, he considers money supply, credit conditions, and the price of oil. He finds that money, credit, and oil probably don’t cause the bulk of economic fluctuations, but do make significant contributions. Because of their aggregate nature, these factors likely affect many firms simultaneously, so their contributions to aggregate volatility should be measurable in the pairwise covariances. I follow Cochrane (1994) in using M2 for money supply, the federal funds rate for credit conditions, and the producer price index for crude petroleum to identify these aggregate effects.

**Industry Factors.** Firm fixed effects capture the impact of industry and other time-invariant firm-specific characteristics on firm-aggregate covariance. Industry effects might arise for a few reasons: First, industry-specific shocks generate higher covariance between firm pairs within an industry in the obvious way. Second, network effects may generate pairwise covariances. A shock to an individual industry can propagate outward from that industry to “connected” industries through the input-output network, where the propagation may run from supplier to customer (Acemoglu et al., 2012) or from customer to supplier (Kelly et al., 2013). Unfortunately, customer-supplier networks are not captured by Compustat, so network effects not captured by firm fixed effects will show up in the error term.

**Other Controls.** While the firm-level fixed effects control for time-invariant firm-level characteristics, I also explicitly control for two time-varying firm characteristics: size, and age. I control for these traits because large firms, and more mature firms, are known to differ systematically from
Table 3 – Covariance Regressed on Explanatory Variables. Table (a) shows regression results for firm-aggregate covariance in productivity, sales, and profit, on a set of explanatory variables. Table (b) shows firm-aggregate covariance over-value regressed on the same explanatory variables. The variable of interest is Number of segments. Controls for financial strength: The quick ratio measures liquidity, debt to capital measures leverage. Controls for aggregate shocks: the M2 measure of the money supply and the West Texas Intermediate spot crude oil price. Other controls: Compustat age is the number of years a firm is listed in Compustat, and employment size is simply the reported employee count. I also include firm-specific fixed effects, which capture industry.

(a) Firm-Aggregate Covariance on Explanatory Variables

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<tr>
<th></th>
<th>X = Sales Growth</th>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>Number of Segments</td>
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<td>0.023***</td>
<td>0.020***</td>
</tr>
<tr>
<td></td>
<td>(0.005)</td>
<td>(0.006)</td>
<td>(0.005)</td>
</tr>
<tr>
<td>Debt to book equity</td>
<td>0.003</td>
<td>0.008</td>
<td>0.005</td>
</tr>
<tr>
<td></td>
<td>(0.015)</td>
<td>(0.017)</td>
<td>(0.016)</td>
</tr>
<tr>
<td>Quick ratio</td>
<td>−0.004</td>
<td>0.001</td>
<td>−0.002</td>
</tr>
<tr>
<td></td>
<td>(0.006)</td>
<td>(0.007)</td>
<td>(0.006)</td>
</tr>
<tr>
<td>Real money growth</td>
<td>−0.001</td>
<td>−0.002</td>
<td>0.002</td>
</tr>
<tr>
<td></td>
<td>(0.004)</td>
<td>(0.004)</td>
<td>(0.004)</td>
</tr>
<tr>
<td>Real oil price growth</td>
<td>0.004</td>
<td>0.001</td>
<td>0.005</td>
</tr>
<tr>
<td></td>
<td>(0.003)</td>
<td>(0.004)</td>
<td>(0.004)</td>
</tr>
<tr>
<td>Years in Compustat</td>
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<td>−0.047***</td>
<td>0.018***</td>
</tr>
<tr>
<td></td>
<td>(0.006)</td>
<td>(0.007)</td>
<td>(0.006)</td>
</tr>
<tr>
<td>Employment size</td>
<td>0.101***</td>
<td>0.112***</td>
<td>−0.039***</td>
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<tr>
<td></td>
<td>(0.006)</td>
<td>(0.006)</td>
<td>(0.006)</td>
</tr>
<tr>
<td>R-squared</td>
<td>0.460</td>
<td>0.333</td>
<td>0.401</td>
</tr>
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</table>

(b) Firm-Aggregate Covariance over Market Value on Explanatory Variables

<table>
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<tr>
<th></th>
<th>X = Sales Growth</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Segments</td>
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<td>−0.025***</td>
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<td></td>
<td>(0.005)</td>
<td>(0.005)</td>
<td>(0.007)</td>
</tr>
<tr>
<td>Debt to book equity</td>
<td>0.059***</td>
<td>0.036*</td>
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<td></td>
<td>(0.014)</td>
<td>(0.014)</td>
<td>(0.020)</td>
</tr>
<tr>
<td>Quick ratio</td>
<td>−0.033***</td>
<td>−0.009</td>
<td>−0.009</td>
</tr>
<tr>
<td></td>
<td>(0.006)</td>
<td>(0.006)</td>
<td>(0.008)</td>
</tr>
<tr>
<td>Real money growth</td>
<td>0.024***</td>
<td>0.004</td>
<td>0.009</td>
</tr>
<tr>
<td></td>
<td>(0.003)</td>
<td>(0.003)</td>
<td>(0.005)</td>
</tr>
<tr>
<td>Real oil price growth</td>
<td>−0.009**</td>
<td>0.002</td>
<td>0.003</td>
</tr>
<tr>
<td></td>
<td>(0.003)</td>
<td>(0.003)</td>
<td>(0.004)</td>
</tr>
<tr>
<td>Years in Compustat</td>
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<td>−0.011*</td>
<td>−0.011</td>
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<tr>
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<td>(0.006)</td>
<td>(0.006)</td>
<td>(0.008)</td>
</tr>
<tr>
<td>Employment size</td>
<td>0.005</td>
<td>0.001</td>
<td>0.002</td>
</tr>
<tr>
<td></td>
<td>(0.005)</td>
<td>(0.005)</td>
<td>(0.008)</td>
</tr>
<tr>
<td>R-squared</td>
<td>0.511</td>
<td>0.528</td>
<td>0.027</td>
</tr>
</tbody>
</table>

The results of the regressions are reported in Table 3. The results lend support to hypothesis (1) for sales, profit, and productivity growth, but effect sizes are smaller for the diversification variable than for the age and size control variables. In general, older and smaller firms covary their smaller, younger counterparts.
less with aggregate growth rates. Neither the financial control variables for leverage and liquidity, nor the aggregate control variables for money, credit, and oil were significant. Results were less significant for hypothesis (2). The coefficients on the diversification variable have a negative sign as the model predicts, but significance at the 0.1% level is only for covariance between firm and aggregate productivity growth. The diversification variable achieves 5% significance for firm-aggregate covariance of sales growth, and is not statistically for profit growth. On the other hand, covariance-over-value for sales growth is significantly explained by the financial and macro factors. For the financial factors, leverage appears to increase covariance-over-value, ceteris paribus, while liquidity appears to decrease it. On the macro side, covariance-over-value increases in money growth, and decreases in the price of oil. The mixed evidence on the diversification effect is likely due to the coarseness of the diversification measure, or to the considerable noise in the rolling-window covariance estimates. More sophisticated methods of measuring the cross-sectional covariance structure will help test the diversification hypothesis. One promising method in the context of large $N$, small $T$ datasets like Compustat is Pesaran (2006)’s panel model with a general multi-factor error structure. The empirical error structure under Pesaran’s approach would find a natural theoretical counterpart in the $\epsilon_{S_t|\mu}$’s of my model. I pursue this idea in a working paper still in early stages of development.

6 Conclusion

I argue that when many firms choose similar risks, their economic fortunes all hang on the same events, and rise and fall together. This comovement of individual outcomes creates aggregate fluctuations, and the chosen risks become systemic. To motivate this interpretation, I document four patterns in the comovement of firm-level activity for a large panel of publicly-traded firms in the United States over the last half-century, and develop a model economy that produces the patterns endogenously by the mechanism just described. My contributions build on recent work on the microeconomic origins of aggregate fluctuations, on financial risk in production economies, and on endogenous fluctuations in macroeconomic models.

My empirical work using Compustat data highlights the pervasiveness of firm-level covariance
in all stages of value creation, and in most periods. I provide evidence for growth rates of three variables: estimated total factor productivity, net sales, and operating income before depreciation. Variance in aggregate growth is driven by pairwise covariance in firm-level growth rates: In most years, pairwise covariances account for 80–90% of aggregate variance. High-productivity firms contribute over six times what the average firm contributes, but less than one-third per dollar of market value.

In my theoretical work, I endogenize covariance by allowing heterogeneous firms to choose risky business lines for themselves, from a menu that I specify exogenously. When firms choose similar technologies, their productivities covary. The model in section 3 endogenously generates aggregate uncertainty from microeconomic shocks, captures qualitative features of the cross-section of stock returns, matches a host of stylized facts related to firm-level volatility and co-movement, and captures empirical regularities related to the number of technologies, product lines, and business segments firms operate. Yet the model’s mechanism is simple: firms choose their technologies.

As a plausibility check on the model’s predictions, I regress the rolling-window firm-aggregate covariances for productivity, sales, and profit growth on the number of business segments firms report in Compustat, controlling for some common explanations of covariance in the literature: money, credit, and oil shocks, firm financial strength, as well as fixed firm characteristics like industry, and time-varying ones like size and age. I find tentative support for the model’s main predictions.

This work suggests new avenues of inquiry. The first is examining covariance structure for small firms. In the present paper, I examine the upper tail of the firm size distribution, as Compustat excludes most small and all private firms. With access to administrative micro datasets, a similar exercise could be carried out for small firms. Second: entry, exit, and dynamic aspects of the technology adoption and abandonment could be studied in an environment with endogenous uncertainty. When firms enter or exit, or change their technology sets, these activities can impact co-movement and volatility over time and in the cross-section in ways that magnify aggregate fluctuations and systemic risk. I introduce these features in an ongoing study of dynamics in a model related to the model I study here. There is also scope to more robustly characterize cross-sectional covariance of firm-level activity in large N small T panels like Compustat. Here I have relied on
A Appendix

In this appendix I provide additional discussion on several aspects of the paper. I start with a mathematical discussion of the model. The discussion includes first-order conditions for the decision problems of the representative household and of consumption goods producers, and derivations of the propositions presented in section 4. I then provide more details on the Compustat dataset and rolling-window covariance calculations used in Figure 1 and described in subsection 5.3. I then turn to the regression analysis presented in subsection 5.4 and provide tables with additional specifications and alternative control variables.

A.1 Mathematical discussion of the model

A.1.1 First-order conditions

Optimality conditions for consumption goods producers. Consider firm $\omega$’s profit maximization problem (5). Eliminate constraints by using (3) and (14) to substitute for $p_{t,v}(\omega)$ and $y_{t,v}(\omega)$ in the firm-vintage profit function (4) that appears in (5). To get first-order optimality conditions, equate with zero the first derivatives of $\Pi_t(\omega)$ with respect to choice variables $k_{t,v}(\omega)$ and $l_{t,v}(\omega)$ for arbitrary vintage $v$. Firm $\omega$’s optimal choice of capital for production with vintage $v$ satisfies

$$k_{t,v}(\omega) = (\alpha) \left( \frac{\theta - 1}{\theta} \right) (Y_t)^{1/\theta} [y_{t,v}(\omega)]^{\theta - 1} (r_t)^{-1}. \quad (34)$$

Its optimal choice of labor satisfies

$$l_{t,v}(\omega) = (1 - \alpha) \left( \frac{\theta - 1}{\theta} \right) (Y_t)^{1/\theta} [y_{t,v}(\omega)]^{\theta - 1} (w_t)^{-1}. \quad (35)$$

Notice that the optimal capital-labor ratio depends neither on the individual firm nor on the vintage of technology:

$$\frac{k_{t,v}(\omega)}{l_{t,v}(\omega)} = \left( \frac{\alpha}{1 - \alpha} \right) \left( \frac{w_t}{r_t} \right). \quad (36)$$

FOCs: Representative household. Now consider the household problem in (11). Eliminate
constraint (9) by using (9) to substitute for $I_t$ in (10). Use the method of Lagrangian multipliers to rewrite the objective function as

$$
\mathcal{L} = \mathbb{E} \left[ \sum_{s=t}^{\infty} \beta^{s-t} u(C_s) - \beta^{s-t} \lambda_s \left( C_s + K_{s+1} + \int_{\omega \in \Omega} V_s(\omega) S_{s+1}(\omega) \lambda(d\omega) \omega \right)
- w_s L - (1 + r_s - \delta) K_s - \int_{\omega \in \Omega} \left[ V_s(\omega) - \Pi_s(\omega) \right] S_s(\omega) \lambda(d\omega) \omega \right] .
$$

To get first-order optimality conditions, equate with zero the first derivatives of $\mathcal{L}$ with respect to choice variables $C_s$, $K_{s+1}$, $S_{s+1}(\omega)$, and $\lambda_s$ for arbitrary period $s$ and firm $\omega$. The household’s optimal plans for consumption, capital accumulation, and equity shares, respectively, satisfy the conditions

$$
\mathbb{E} \left[ u'(C_s) \right] = \mathbb{E} \left[ \lambda_s \right],
$$

$$
\mathbb{E} \left[ \lambda_s \right] = \beta \mathbb{E} \left[ \lambda_{s+1} (1 + r_{s+1} - \delta) \right],
$$

$$
\mathbb{E} \left[ \lambda_s V_s(\omega) \right] = \beta \mathbb{E} \left[ \lambda_{s+1} (V_{s+1}(\omega) + \Pi_{s+1}(\omega)) \right].
$$

Now derive the household’s stochastic discount factor from these conditions. To start, set $s = t$ and use (38) and (40) to write firm $\omega$’s period-$t$ present value as

$$
V_t(\omega) = \mathbb{E} \left[ \left( \frac{\partial u'(C_{t+1})}{u'(C_t)} \right) (V_{t+1}(\omega) + \Pi_{t+1}(\omega)) \right].
$$

The household’s one-period stochastic discount factor is the first term in the expectation operator:

$$
m_{t,t+1} = \beta u'(C_{t+1})/u'(C_t) .
$$

Iterate (41) via $V_{t+1}(\omega)$ to get the multi-period stochastic discount factor.
factor. For any period \( s \geq t \), write the latter as

\[
m_{t,s} = m_{t,t+1} \cdot m_{t+1,t+2} \cdot \ldots \cdot m_{s-1,s} = \beta^s \frac{u'(C_{s+1})}{u'(C_t)} \frac{u'(C_{s+2})}{u'(C_{t+1})} \ldots \frac{u'(C_s)}{u'(C_{t+1})}.
\] (42)

Optimality conditions for capital goods producers. Now consider the profit maximization problem in the (15). Using (??) in (??) and taking derivatives with respect to factors,

\[
r_t = \alpha Z_t(k_t)^{\alpha-1}(l_t)^{1-\alpha},
\] (43)

\[
w_t = (1 - \alpha)Z_t(k_t)^{\alpha}(l_t)^{-\alpha},
\] (44)

and notice that the capital-labor ratio in the capital goods sector is again

\[
\frac{k_t}{l_t} = \left( \frac{\alpha}{1 - \alpha} \right) \left( \frac{w_t}{r_t} \right).
\] (45)

A.1.2 Main propositions and proofs

Proposition 4.1 A productivity aggregate over technologies summarizes all of the technological heterogeneity within an individual firm \( \omega \):

\[
Z_t(\omega) = \left[ \int_{V(\omega)} [z(\omega)z_{t,v}]^{\beta-1} \lambda(dv) \right]^{\frac{1}{\beta-1}}.
\] (46)

A productivity aggregate over firms summarizes all of the firm-specific and technological hetero-
geneity within the consumption goods sector:

\[
Z_t = \left[ \int_{\Omega} Z_t(\omega)^{\theta - 1} \lambda(d\omega) \right]^{\frac{1}{\theta - 1}}.
\]

(47)

Aggregate factor demands, production, and profit can be written in terms of aggregate productivities and variables that do not vary across entities.

**Proof.** The household and capital goods producer are representative, so aggregation pertains only to the final goods sector.

Start with the optimality conditions (34) and (35) from the firm’s decision problem (5). These expressions contain vintage-specific variables \( k_{t,v}(\omega) \), \( l_{t,v}(\omega) \), and \( y_{t,v}(\omega) \) as well as variables and parameters common to all vintages. Combine equations (34) and (35) with the production function (3) to obtain expressions for \( k_{t,v}(\omega) \), \( l_{t,v}(\omega) \), and \( y_{t,v}(\omega) \) in terms of \( z_{t,v} \) and variables and parameters common to all vintages:

\[
k_{t,v}(\omega) = [z(\omega) z_{t,v}]^{\theta - 1} (Y_t) \left( \frac{\theta - 1}{\theta} \right)^{\theta} \left( \frac{r_t}{\alpha} \right)^{\alpha(1-\theta) - 1} \left( \frac{w_t}{1 - \alpha} \right)^{(1-\alpha)(1-\theta)},
\]

(48)

\[
l_{t,v}(\omega) = [z(\omega) z_{t,v}]^{\theta - 1} (Y_t) \left( \frac{\theta - 1}{\theta} \right)^{\theta} \left( \frac{r_t}{\alpha} \right)^{\alpha(1-\theta) - 1} \left( \frac{w_t}{1 - \alpha} \right)^{(1-\alpha)(1-\theta) - 1},
\]

(49)

\[
y_{t,v}(\omega) = [z(\omega) z_{t,v}]^{\theta} (Y_t) \left( \frac{\theta - 1}{\theta} \right)^{\theta} \left( \frac{r_t}{\alpha} \right)^{-\alpha \theta} \left( \frac{w_t}{1 - \alpha} \right)^{-(1-\alpha)\theta}.
\]

(50)

These expressions can be simplified further using an expression derived from the definition of
the consumption basket, along with (50) and market clearing:

\[
Y_t = \left[ \int_{\Omega} \int_{V(\omega)} \left[ y_{t,v}(\omega) \right]^{\frac{1}{\theta} - 1} \lambda (dvd\omega) \right]^{\theta}
\]

\[
= \left( \frac{\theta - 1}{\theta} \right)^{\theta} \left( \frac{\alpha}{r_t} \right)^{\alpha \theta} \left( \frac{1 - \alpha}{w_t} \right)^{(1-\alpha)\theta} \left( Y_t \right)^{\frac{1}{\theta} - 1} \left[ \int_{\Omega} \int_{V(\omega)} (z(\omega)z_{t,v})^{\theta - 1} \lambda (dvd\omega) \right]^{\theta}
\]

\[
\Leftrightarrow Z_t := \left[ \int_{\Omega} \int_{V(\omega)} (z(\omega)z_{t,v})^{\theta - 1} \lambda (dvd\omega) \right]^{\frac{1}{\theta}} = \left( \frac{\theta}{\theta - 1} \right) \left( \frac{r_t}{\alpha} \right)^{\alpha} \left( \frac{w_t}{1 - \alpha} \right)^{1 - \alpha}.
\]

Now use the expression for \( Z_t \) to simplify (48)–(50):

\[
k_{t,v}(\omega) = \left( \frac{\theta - 1}{\theta} \right) \left( \frac{\alpha}{r_t} \right) \left( \frac{z(\omega)z_{t,v}}{Z_t} \right)^{\theta - 1} Y_t
\]

\[
l_{t,v}(\omega) = \left( \frac{\theta - 1}{\theta} \right) \left( \frac{1 - \alpha}{w_t} \right) \left( \frac{z(\omega)z_{t,v}}{Z_t} \right)^{\theta - 1} Y_t
\]

\[
y_{t,v}(\omega) = \left( \frac{z(\omega)z_{t,v}}{Z_t} \right)^{\theta} Y_t.
\]

Now recall that \( p_{t,v}(\omega) = (y_{t,v}(\omega)/Y_t)^{-(1/\theta)} \), and use above to get a similar expression for profit:

\[
\pi_{t,v}(\omega) = p_{t,v}(\omega)y_{t,v}(\omega) - r_t k_{t,v}(\omega) - w_t l_{t,v}(\omega)
\]

\[
= \frac{1}{\theta} \left( \frac{z(\omega)z_{t,v}}{Z_t} \right)^{\theta - 1} Y_t.
\]

To get firm aggregates, sum the \( k_{t,v}(\omega) \)'s, \( l_{t,v}(\omega) \)'s, and \( \pi_{t,v}(\omega) \)'s, and use the Dixit-Stiglitz
aggregator on \( y_{t,v}(\omega) \):

\[
K_t(\omega) := \int_{\nu(\omega)} k_{t,v}(\omega) \lambda(dv) = \left( \frac{\theta - 1}{\theta} \right) \left( \frac{\alpha}{w_t} \right) \left( \frac{Z_t(\omega)}{Z_t} \right)^{\theta-1} Y_t
\]

\[
L_t(\omega) := \int_{\nu(\omega)} l_{t,v}(\omega) \lambda(dv) = \left( \frac{\theta - 1}{\theta} \right) \left( \frac{1 - \alpha}{w_t} \right) \left( \frac{Z_t(\omega)}{Z_t} \right)^{\theta-1} Y_t
\]

\[
Y_t(\omega) := \int_{\nu(\omega)} (y_{t,v}(\omega))^{\frac{\theta-1}{\theta}} \lambda(dv) \underset{\nu(\omega)}{\overset{\varphi}{\sim}} \left( \frac{Z_t(\omega)}{Z_t} \right)^{\theta} Y_t
\]

\[
\Pi_t(\omega) := \int_{\nu(\omega)} \pi_{t,v}(\omega) \lambda(dv) = \frac{1}{\theta} \left( \frac{Z_t(\omega)}{Z_t} \right)^{\theta-1} Y_t,
\]

where

\[
Z_t(\omega) := \left[ \int_{\nu(\omega)} (z(\omega) z_{t,v})^{\theta-1} \lambda(dv) \right]^{\frac{1}{\theta-1}}.
\]

Further rearrangement along the same lines yields the economy-wide aggregates, I will type the derivations up properly soon. It is also possible to write aggregate output in terms of a Cobb-Douglas aggregate production function, at both the firm and economy-wide levels. I still need to type these results up, they’re in notes dated 090817. The result of the derivations is:

\[
Y_t(\omega) = Z_t(\omega) [K_t(\omega)]^\alpha [L_t(\omega)]^{1-\alpha}
\]

\[
Y_t = Z_t(K_t)^\alpha (L_t)^{1-\alpha},
\]

where the production function arguments should be understood as optimal factor inputs that satisfy the firm’s optimality conditions for from the profit maximization problem.

These aggregate production functions look familiar from simpler production structures, so this model looks homomorphic to those simpler models. But remember that shocks here are endogenous, so the models would only be homomorphic if the stochastic properties of random productivity in
the simpler models were restricted to match the endogenous aggregates in this model.

Notice that if technology sets \( \mathcal{V}(\omega) \) differs across firms, so too will the distributions of the random productivity variables.

**Proposition 4.2** In a non-stochastic steady state, any firm \( \omega \) with productivity \( z(\omega) \geq \bar{z} \) chooses technology set \( \mathcal{V}(\omega) = \{ v \in \mathcal{V} : v \leq v \leq \bar{\pi}(\omega) \} \), where the endogenous cut-offs \( \bar{z} \) and \( \bar{\pi}(\omega) \) are given by:

\[
\bar{z} = \left( \frac{\theta}{\mu} \right)^{\frac{1}{\gamma}} \tag{53}
\]

\[
\bar{\pi}(\omega) = \left( \frac{\mu}{\theta} \right)^{\frac{1}{\gamma}} z(\omega)^{\frac{\theta - 1}{\gamma}}. \tag{54}
\]

The above cut-offs are also first-order approximate to those that obtain in a stochastic world. Firms with \( z(\omega) < \bar{z} \) do not produce. Under parameter restrictions, firms \( \omega_1 \) and \( \omega_2 \) with productivities \( \bar{z} < z(\omega_1) < z(\omega_2) \) choose technology sets such that \( \mathcal{V}(\omega_1) \subset \mathcal{V}(\omega_2) \).

**Proof.** Firms choose their technology sets \( \mathcal{V}(\omega) \subseteq \mathcal{V} = [v, \infty) \subseteq \mathbb{R}^+ \) to maximize profit. Recall from (??) that technologies differ in their period fixed costs, but not their first two moments. Starting from the technology adoption rule in (7), and rearranging:

\[
E_t [m_{t,t+1}(\pi_{t,v}(\omega) - f_{s,v})] > 0
\]

\[
E_t \left[ \frac{\beta u'(C_{t+1})}{u'(C_t)} (\pi_{t,v}(\omega) - f_{s,v}) \right] > 0
\]

\[
E_t \left[ \frac{1}{\gamma t+1} (\pi_{t+1,v}(\omega) - f_{t+1,v}) \right] > 0,
\]

where the third line assumes log utility. Now recall:

\[
\pi_{t,v}(\omega) = \frac{1}{\theta} \left( \frac{z(\omega)z_{t,v}}{Z_t} \right)^{\theta - 1} Y_t
\]

\[
f_{t,v} = \frac{Y_t}{\mu} v^\gamma.
\]
Using these expressions in the adoption rule:

\[
E_t \left[ \frac{1}{Y_{t+1}} \left( \pi_{t+1,v}(\omega) - f_{t+1,v} \right) \right] > 0
\]

\[\Leftrightarrow \left( \frac{z(\omega)^{\theta - 1}}{\theta} \right) E_t \left[ \left( \frac{z_{t,v} Z_t}{Z_t} \right)^{\theta - 1} \right] \geq \frac{v^\gamma}{\mu}. \]

From here, either evaluate the productivities in the ratio under the expectation operator at their expected values to get an expression describing steady-state technology sets, or take an approximation of the expression under the expectation operator. A first-order approximation gives the same results as the steady-state solution:

\[\tau(\omega) = \left( \frac{\mu e}{\theta} \right)^{\frac{1}{\gamma}} z(\omega)^{\frac{\theta - 1}{\gamma}} \]

\[\Rightarrow z(v) = \left( \frac{\mu e}{\theta} \right)^{\frac{1}{\gamma - 1}} v^{\frac{\gamma}{\gamma - 1}}. \]

Notice that the cut-off \(\tau(\omega)\) increasing in \(z(\omega)\), so the more productive firms produce more varieties and use more technology.

Two quick remarks: First, it’s useful that the steady-state and first-order approximate cut-offs coincide, because it means that first-order dynamics around the steady state are completely standard in this model. In a companion paper, I pursue a dynamic version of the technology adoption problem in which first-order approximate technology sets differ from their steady-state counterparts because state variables appear in the adoption rule. In that model, uncertainty varies with the state of the economy.

Second, I’ve looked at the second-order approximate case, and it gives more interesting but less tractable results. There is a covariance term in the second-order approximation that varies with \(v\)—covariance is higher for commonly-used technologies. I have notes on this case, can share if you’re interested.

**Proposition 4.3** Let technology sets be those that firms choose in the non-stochastic steady state.
Then the first and second moments of firm-level productivity, denoted $\mu(\omega) = E_t [Z_t(\omega)^{\theta-1}[t+1]]$ and $\sigma^2(\omega) = \text{Var}_t (Z_t(\omega)^{\theta-1}[t+1])$, are given by:

$$
\mu(\omega) = \mu_{\epsilon} z(\omega)^{\theta-1} \left( \frac{z(\omega)}{z} \right)^{\zeta_{\omega}} - 1 \quad \text{(55)}
$$

$$
\sigma^2(\omega) = \sigma^2_{\epsilon} z(\omega)^{2(\theta-1)} \left( \frac{z(\omega)}{z} \right)^{\zeta_{\omega}} - 1 \quad \text{(56)}
$$

The first and second moments of aggregate productivity, denoted $\mu = E [Z_t^{\theta-1}]$ and $\sigma^2 = \text{Var} (Z_t^{\theta-1})$, respectively, are given by:

$$
\mu = \mu_{\epsilon} \theta z^{\theta-1} \quad \text{(57)}
$$

$$
\sigma^2 = \sigma^2_{\epsilon} \zeta z^{2(\kappa-(\theta-1))} \quad \text{(58)}
$$

Under parameter restrictions, the first and second moments of all productivity aggregates are positive and finite, and for any firms $\omega_1$ and $\omega_2$ with $z(\omega_1) < z(\omega_2)$, it holds that $\mu_t(\omega_1) < \mu_t(\omega_2)$ and $\sigma_t^2(\omega_1) < \sigma_t^2(\omega_2)$.

**Proof.** Begin with the first moment of sector-aggregate productivity, just using the definition:

$$
\mu = E [Z_t^{\theta-1}] = E \left[ \int_{\Omega} Z_t(\omega)^{\theta-1} \lambda(\omega) \right] \\
= E \left[ \int_{\Omega} \int_{V(\omega)} (z(\omega)z_{t,v})^{\theta-1} \lambda(dvd\omega) \right] \\
= E \left[ \int_{V} \int_{\Omega_v} (z(\omega)z_{t,v})^{\theta-1} \lambda(d\omega dv) \right] \\
= E \left[ \int_{V} z_{t,v}^{\theta-1} \left( \int_{\Omega_v} z(\omega)^{\theta-1} \lambda(d\omega) \right) \lambda(d\omega) \right],
$$

49
where $\Omega_v$ is the set of firms using vintage $v$, that is: $\Omega_v := \{ \omega \in \Omega : z(\omega) < z(v) \}$, and $z(v)$ is the inverse of the cost cut-off $\pi(v)$.

Now evaluate the inner integral:

$$\int_{\Omega_v} z(\omega)^{\theta-1} \lambda(d\omega) = \int_{\Omega} z(\omega)^{\theta-1} h(z(\omega)) dz(\omega)$$

$$= \left[ \frac{\kappa}{(\theta - 1) - \kappa} z(\omega)^{(\theta-1)-\kappa} \right]_{\Omega}^{\infty} z(v)^{(\theta-1)-\kappa}$$

$$= \left( \frac{\kappa}{\kappa - (\theta - 1)} \right) z(v)^{(\theta-1)-\kappa}$$

$$= \left( \frac{\kappa}{\kappa - (\theta - 1)} \right) \left( \frac{\mu_e}{\theta} \right)^{\theta-1} \left( \frac{1}{v} \right)^{\frac{\gamma(n-(\theta-1))}{\theta+1}}.$$

Substitute the evaluated integral back into the expression for $\mu$:

$$\mu = E \left[ Z_t^{\theta-1} \right] = \left( \frac{\kappa}{\kappa - (\theta - 1)} \right) \left( \frac{\mu_e}{\theta} \right)^{\theta-1} \int_{\Omega} z(\omega)^{\theta-1} h(z(\omega)) dz(\omega)$$

$$= \left( \frac{\kappa}{\kappa - (\theta - 1)} \right) \left( \frac{\mu_e}{\theta} \right)^{\theta-1} E \left[ \int_{\Omega} z(\omega)^{\theta-1} h(z(\omega)) dz(\omega) \right]$$

Use the definition of technological productivity $z_{t,v} := \epsilon_{t,[v]}$, set $v = 1$, and write the remaining integral as:

$$E \left[ \int_{\Omega} z_{t,v}^{\theta-1} \epsilon_{t,[v]}^{\frac{\gamma(n-(\theta-1))}{\theta+1}} \lambda(dv) \right] = E \left[ \int_{\Omega} \epsilon_{t,[v]}^{\theta-1} \epsilon_{t,[v]}^{\frac{\gamma(n-(\theta-1))}{\theta+1}} \lambda(dv) \right]$$

$$= E \left[ \epsilon_{t,2}^{\theta-1} \epsilon_{t,2}^{\frac{\gamma(n-(\theta-1))}{\theta+1}} \lambda(dv) + \epsilon_{t,3}^{\theta-1} \epsilon_{t,3}^{\frac{\gamma(n-(\theta-1))}{\theta+1}} \lambda(dv) + \ldots \right]$$

$$= \mu_e \epsilon_{t,1}^{\frac{\gamma(n-(\theta-1))}{\theta+1}} \lambda(dv) + \mu_e \epsilon_{t,2}^{\frac{\gamma(n-(\theta-1))}{\theta+1}} \lambda(dv) + \ldots.$$
Now consider the integrals of the form:

\[
\int_{n}^{n+1} \frac{\gamma^{n-(\theta-1)}}{\gamma \kappa - (\theta-1)} \lambda(dv) = \left[ \frac{\theta - 1}{\gamma \kappa - (\theta-1)} \right]^{n+1} - \left[ \frac{\theta - 1}{\gamma \kappa - (\theta-1)} \right]^{n}
\]

Returning to the expression for \( \mu \):

\[
\mu = E \left[ Z_t^{\theta-1} \right] = \left( \frac{\kappa}{\kappa - (\theta-1)} \right) \left( \frac{\mu_e}{\theta} \right) \mu_e \sum_{n=1}^{\infty} \left[ \left( \frac{\theta - 1}{\gamma \kappa - (\theta-1)} \right) - \left( \frac{\theta - 1}{\gamma \kappa - (\theta-1)} \right) \right] \times \left[ \left( \frac{1}{n} \right) \gamma^{n-(\theta-1)} - \left( \frac{1}{n+1} \right) \gamma^{n-(\theta-1)} \right]
\]

Now turn to the second moment of sector-aggregate productivity. Starting again with the definition:

\[
\sigma^2 = \text{Var} \left( Z_t^{\theta-1} \right) = \text{Var} \left( \int_{\Omega} Z_t(\omega)^{\theta-1} \lambda(d\omega) \right)
\]

\[
= \text{Var} \left( \int_{\Omega} \int_{\Omega(\omega)} (z(\omega)z_{t,v})^{\theta-1} \lambda(dvd\omega) \right)
\]

\[
= \text{Var} \left( \int_{\Omega} z_{t,v}^{\theta-1} \int_{\Omega(\omega)} z(\omega)^{\theta-1} \lambda(d\omega)dv \right)
\]

\[
= \text{Var} \left( \int_{\Omega} \int_{\Omega(\omega)} z(\omega)^{\theta-1} \lambda(d\omega)dv \right)
\]

\[
= \text{Var} \left( \int_{\Omega} \int_{\Omega(\omega)} z(\omega)^{\theta-1} \lambda(d\omega)dv \right)
\]

where from the third to the fourth line I change measure from Lebesgue to Pareto. Continuing,
\[ \int_{\psi} z_t^\theta v^{-\gamma|\kappa-(\theta-1)|\theta-1} \lambda(\omega) = \int_{\psi} \epsilon_{t,[v]} v^{-\gamma|\kappa-(\theta-1)|\theta-1} \lambda(\omega) \]

\[ \quad = \sum_{n=1}^{\infty} \frac{\epsilon_{t,n+1}(\theta-1)}{\gamma[\kappa-(\theta-1)]-(\theta-1)} \left[ \frac{1}{n} \gamma_n^\theta(\theta-1) \right] \left[ \frac{1}{n} \gamma_n^\theta(\theta-1) \right] - \left( \frac{1}{n+1} \right) \gamma_n^\theta(\theta-1) \]

\[ \sigma^2 = \text{Var} \left( Z_t^{\theta-1} \right) = \left( \frac{\kappa}{\kappa-(\theta-1)} \right)^2 \left( \frac{\theta}{\mu_e} \right)^{-2 \frac{\kappa-(\theta-1)}{\theta-1}} \times \text{Var} \left( \sum_{n=1}^{\infty} \frac{\epsilon_{t,n+1}(\theta-1)}{\gamma[\kappa-(\theta-1)]-(\theta-1)} \left[ \frac{1}{n} \gamma_n^\theta(\theta-1) \right] \left[ \frac{1}{n} \gamma_n^\theta(\theta-1) \right] - \left( \frac{1}{n+1} \right) \gamma_n^\theta(\theta-1) \right) \]

\[ \quad = \sigma_t^2 \left( \frac{\kappa}{\kappa-(\theta-1)} \right)^2 \left( \frac{\theta}{\mu_e} \right)^{-2 \frac{\kappa-(\theta-1)}{\theta-1}} \times \sum_{n=1}^{\infty} \left[ \frac{1}{n} \gamma_n^\theta(\theta-1) \right] \left[ \frac{1}{n+1} \right] \gamma_n^\theta(\theta-1)^2 \right] \]

\[ \quad \times \left[ \frac{1}{n} \gamma_n^\theta(\theta-1) \right] \left[ \frac{1}{n+1} \right] \gamma_n^\theta(\theta-1)^2 \right] \]
Notice that $\theta/\mu$ appears on the right-hand side. Substituting it for $z$, and collecting parameters,

$$
\sigma^2 = \text{Var} \left( Z^{\theta-1}_t \right) = \sigma^2 \cdot \zeta \cdot z^{-2[\kappa-(\theta-1)]},
$$

where

$$
\zeta := \left( \frac{\kappa}{\kappa - (\theta - 1)} \right)^2 \left( \frac{(\theta - 1)}{\gamma [\kappa - (\theta - 1)] - (\theta - 1)} \right)^2 \sum_{n=1}^{\infty} \left[ \left( \frac{1}{n} \right)^{\gamma [\kappa - (\theta - 1)] - (\theta - 1)} - \left( \frac{1}{n + 1} \right)^{\gamma [\kappa - (\theta - 1)] - (\theta - 1)} \right]^2.
$$

**Proposition 4.4** Let technology sets be those that firms choose in the non-stochastic steady state. Then the covariance between firm and aggregate productivity, denoted by $\sigma_{\omega,\Omega}(\omega) = \text{Cov} \left( Z^{\theta-1}_t(\omega), Z^{\theta-1}_t \right)$, is given by

$$
\sigma_{\omega,\Omega}(\omega) = z(\omega)^{\theta-1} \zeta \left[ 1 - \left( \frac{\hat{z}}{z(\omega)} \right)^{\zeta_{\Omega,2}} \right]
$$

The covariance between firm and aggregate productivity, expressed as a fraction of firm market value, is given by

$$
\frac{\sigma_{\omega,\Omega}(\omega)}{V_t(\omega)} = \frac{1}{V_t} \left( \frac{\zeta_{\Omega,1}}{z(\omega)} \left[ 1 - \left( \frac{\hat{z}}{z(\omega)} \right)^{\zeta_{\Omega,2}} \right] + \zeta_{\Omega,3} \left( \frac{1}{z(\omega)} \right)^{\zeta_{\Omega,4}} - \left( \frac{1}{2} \right)^{\theta-1} \right).
$$

Under parameter restrictions, covariance-over-value falls for all $z(\omega)$ above a threshold. The ratio falls in the level of aggregate output.

**Proof.** To start, identify a specific firm $\omega_1$, use the definitions of $Z_t(\omega_1)$ and $Z_t$ in the covariance
expression, and the cut-offs $z$ and $\pi(\omega)$ for the integral bounds:

$$\sigma_{\omega t}(\omega) = \text{Cov} \left( Z_t(\omega_1)^{\theta - 1}, Z_t^{\theta - 1} \right) = \text{Cov} \left( \int_{\Omega} [z(\omega_1)z_{t,v}]^{\theta - 1} \lambda(d\omega), \int_{\Omega} Z_t(\omega)^{\theta - 1} \lambda(d\omega) \right)$$

$$= \text{Cov} \left( \int_{z=1}^{\pi(\omega)} [z(\omega)z_{t,v}]^{\theta - 1} \lambda(dv), \int_{z=1}^{\infty} [z(\omega)z_{t,v}]^{\theta - 1} \lambda(dvd\omega) \right).$$

Now consider the first integral:

$$\int_{z=1}^{\pi(\omega)} [z(\omega_1)z_{t,v}]^{\theta - 1} \lambda(dv) = z(\omega_1)^{\theta - 1} \int_{z=1}^{\infty} \epsilon_{t,v} \lambda(dv)$$

$$= z(\omega_1)^{\theta - 1} \left[ \int_{v=1}^{2} \epsilon_{t,v} \lambda(dv) + \int_{v=2}^{3} \epsilon_{t,v} \lambda(dv) + \cdots + \int_{v=\pi(\omega)}^{\pi(\omega) - 1} \epsilon_{t,v} \lambda(dv) \right]$$

$$= z(\omega_1)^{\theta - 1} \sum_{n=1}^{\pi(\omega) - 1} \epsilon_{t,n+1},$$

where I have assumed w.l.g. that $\pi(\omega) \in \mathbb{N}$.

Now consider the second integral:

$$\int_{z=1}^{\infty} \int_{z=1}^{\pi(\omega)} [z(\omega)z_{t,v}]^{\theta - 1} \lambda(dvd\omega) = \int_{z=1}^{\infty} z_{t,v}^{\theta - 1} \left( \int_{z(\omega)}^{z(\omega_1)} \left( \int_{z(\omega)}^{\infty} z(\omega)^{\theta - 1} \lambda(d\omega) \right) \lambda(dv) \right)$$

$$= \int_{z=1}^{\infty} z_{t,v}^{\theta - 1} \left( \int_{z(\omega)}^{\infty} z(\omega)^{\theta - 1} h(z(\omega)) \lambda(dz(\omega)) \right) \lambda(dv)$$

$$= \int_{z=1}^{\infty} z_{t,v}^{\theta - 1} \frac{\kappa}{\kappa - (\theta - 1)} z(\omega)^{-(\theta - 1)} \lambda(dv),$$

where line two changes measure from Lebesgue to Pareto. Continuing with the second integral,
using \( z(v) = \left( \frac{\theta \mu}{\mu_e} \right)^{\frac{1}{\gamma - \eta}} v^{\frac{1}{\gamma - \eta}} \),

\[
\int_0^\infty \int_0^1 \left[ z(\omega) z_{t,v} \right]^{\theta - 1} \lambda(dv d\omega) = \left( \frac{\kappa}{\kappa - (\theta - 1)} \right) \left( \frac{\theta \mu}{\mu_e} \right)^{\frac{\gamma [\kappa - (\theta - 1)]}{\eta + 1}} \int_0^\infty z_{t,v}^{\theta - 1} v^{\frac{\gamma [\kappa - (\theta - 1)]}{\eta + 1}} \lambda(dv).
\]

Now the single integral on the right-hand side:

\[
\int_0^\infty z_{t,v}^{\theta - 1} v^{\frac{\gamma [\kappa - (\theta - 1)]}{\eta + 1}} \lambda(dv) = \int_0^\infty \epsilon_{t,v}^{\theta - 1} v^{\frac{\gamma [\kappa - (\theta - 1)]}{\eta + 1}} \lambda(dv)
\]

\[
= \epsilon_{t,2} \int_1^2 v^{\frac{\gamma [\kappa - (\theta - 1)]}{\eta + 1}} \lambda(dv) + \epsilon_{t,3} \int_2^3 v^{\frac{\gamma [\kappa - (\theta - 1)]}{\eta + 1}} \lambda(dv) + \ldots
\]

Now consider the integrals of the form:

\[
\int_0^\infty v^{\frac{\gamma [\kappa - (\theta - 1)]}{\eta + 1}} \lambda(dv) = \left[ \frac{\theta - 1}{\gamma [\kappa - (\theta - 1)] + (\theta - 1)} \right]^{n+1} v^{\frac{\gamma [\kappa - (\theta - 1)]}{\eta + 1}}
\]

\[
= \left( \frac{\theta - 1}{\gamma [\kappa - (\theta - 1)] + (\theta - 1)} \right) \left[ \frac{1}{n} \right]^{\frac{\gamma [\kappa - (\theta - 1)]}{\eta + 1}} - \left( \frac{1}{n+1} \right)^{\frac{\gamma [\kappa - (\theta - 1)]}{\eta + 1}}
\]

So the single integral becomes:

\[
\int_0^\infty z_{t,v}^{\theta - 1} v^{\frac{\gamma [\kappa - (\theta - 1)]}{\eta + 1}} \lambda(dv) = \left( \frac{\theta - 1}{\gamma [\kappa - (\theta - 1)] + (\theta - 1)} \right)
\]

\[
\times \sum_{n=1}^\infty \epsilon_{t,n+1} \left[ \frac{1}{n} \right]^{\frac{\gamma [\kappa - (\theta - 1)]}{\eta + 1}} - \left( \frac{1}{n+1} \right)^{\frac{\gamma [\kappa - (\theta - 1)]}{\eta + 1}}
\]

\[55\]
and the second integral becomes:

\[
\int_0^\infty \int_0^\infty (\omega z_t)^{\theta - 1} \lambda(d\omega) = \left( \frac{\kappa}{\kappa - (\theta - 1)} \right) \left( \frac{\theta \mu}{\mu_{\epsilon}} \right) \left( \frac{\theta - 1}{\gamma[\kappa - (\theta - 1)] + (\theta - 1)} \right) \times \sum_{n=1}^{\infty} \epsilon_{t,n+1} \left[ \left( \frac{1}{n} \right)^{\gamma \frac{\kappa - (\theta - 1)}{\theta - 1}} - \left( \frac{1}{n + 1} \right)^{\gamma \frac{\kappa - (\theta - 1)}{\theta - 1}} \right].
\]

Now, recall that Cov (\(\epsilon_{t,n}, \epsilon_{t,m}\)) = 0 for \(n \neq m\), and write the desired covariance as:

\[
\sigma_{\omega,t}(\omega) = \text{Cov} (Z_t^{\theta - 1}, Z_t^{\theta - 1})
\]

\[
= z(\omega_1)^{\theta - 1} \left( \frac{\kappa}{\kappa - (\theta - 1)} \right) \left( \frac{\theta \mu}{\mu_{\epsilon}} \right) \left( \frac{\theta - 1}{\gamma[\kappa - (\theta - 1)] + (\theta - 1)} \right) \times \text{Cov} \left( \sum_{n=1}^{\infty} \epsilon_{t,n+1}, \sum_{n=1}^{\infty} \epsilon_{t,n+1} \left[ \left( \frac{1}{n} \right)^{\gamma \frac{\kappa - (\theta - 1)}{\theta - 1}} - \left( \frac{1}{n + 1} \right)^{\gamma \frac{\kappa - (\theta - 1)}{\theta - 1}} \right] \right)
\]

\[
= z(\omega_1)^{\theta - 1} \left( \frac{\kappa}{\kappa - (\theta - 1)} \right) \left( \frac{\theta \mu}{\mu_{\epsilon}} \right) \left( \frac{\theta - 1}{\gamma[\kappa - (\theta - 1)] + (\theta - 1)} \right) \times \text{Cov} \left( \epsilon_{t,n+1}, \epsilon_{t,n+1} \right)
\]

\[
= \sigma^2 \left( \frac{\kappa}{\kappa - (\theta - 1)} \right) \left( \frac{\theta \mu}{\mu_{\epsilon}} \right) \left( \frac{\theta - 1}{\gamma[\kappa - (\theta - 1)] + (\theta - 1)} \right) \times \text{Cov} \left( \epsilon_{t,n+1}, \epsilon_{t,n+1} \right)
\]

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Notice that the right-hand side summation, with $a$ as a temporary placeholder, is of form:

$$\sum_{n=1}^{v(\omega_1)-1} \left[ \left( \frac{1}{n} \right)^a - \left( \frac{1}{n+1} \right)^a \right] = \left[ \left( \frac{1}{1} \right)^a - \left( \frac{1}{2} \right)^a + \left( \frac{1}{2} \right)^a - \left( \frac{1}{3} \right)^a + \cdots - \left( \frac{1}{v(\omega_1)} \right)^a \right]$$

$$= \left[ 1 - \left( \frac{1}{v(\omega_1)} \right)^a \right].$$

Returning to the covariance expression, and simplifying the summation as above,

$$\sigma_{\omega_1}(\omega) = \text{Cov} \left( Z_t^\theta, Z_t^\theta \right)$$

$$= \sigma^2 z(\omega_1)^{\theta-1} \left( \frac{\kappa}{\kappa-(\theta-1)} \right) \left( \frac{\theta \mu}{\mu_c} \right)^{-\frac{[\kappa-(\theta-1)]}{\theta-1}} \left( \frac{\theta - 1}{\gamma \left( \kappa - (\theta - 1) \right) + (\theta - 1)} \right) \times \left[ 1 - \left( \frac{1}{v(\omega_1)} \right)^{\frac{\gamma \left( \kappa - (\theta - 1) \right) z(\omega_1)}{\gamma(\theta-1)}} \right]$$

$$= \sigma^2 z(\omega_1)^{\theta-1} \left( \frac{\kappa}{\kappa-(\theta-1)} \right) \left( \frac{\theta \mu}{\mu_c} \right)^{-\frac{[\kappa-(\theta-1)]}{\theta-1}} \left( \frac{\theta - 1}{\gamma \left( \kappa - (\theta - 1) \right) + (\theta - 1)} \right) \times \left[ 1 - \left( \frac{\theta \mu}{\mu_c} \right)^{\frac{\gamma \left( \kappa - (\theta - 1) \right) z(\omega_1)}{\gamma(\theta-1)}} \left( \frac{1}{z(\omega_1)} \right)^{\frac{\gamma \left( \kappa - (\theta - 1) \right)}{\gamma}} \right]$$

where the last line uses $v(\omega_1) = \left( \frac{\mu_c}{\theta \mu} \right)^{\frac{\gamma}{\gamma}} z(\omega_1)^{\frac{\theta-1}{\gamma}}$. 

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Finally, collect parameters, return from specific $\omega_1$ to arbitrary $\omega$, and write:

$$\frac{\sigma_{\omega_1}(\omega)}{\sigma_2^2} = z(\omega)^{\theta-1} \zeta_{\omega_1} \left[ 1 - \left( \frac{z}{\bar{z}(\omega)} \right)^{\zeta_{\omega_1}^2} \right],$$

where

$$\zeta_{\omega_1} = \left( \frac{\kappa}{\kappa - (\theta - 1)} \right) \left( \frac{\theta}{\mu} \right)^{-[\theta-(\theta-1)]} \left[ \frac{\theta - 1}{\gamma \kappa - (\theta - 1) + (\theta - 1)} \right]$$

$$\zeta_{\omega_1} = \frac{\gamma \kappa - (\theta - 1)}{\gamma}.$$

Recall that the $\mu$ appearing in $\zeta_{\omega_1}$ has already been expressed in terms of parameters, so the above expression suffices.

Now turn to covariance over market value. Start from the following primitives:

$$V_t(\omega) = E \left[ \beta^{s-t} \sum_{s=t+1}^{\infty} \frac{u'(C_s)}{u'(C_t)} (\Pi_s(\omega) - F_s(\omega)) \right]$$

$$\Pi_t(\omega) = \int_{V_t(\omega)} \pi_{t,\omega}(\omega) \lambda(\omega) = \frac{1}{\bar{\theta}} \left( \frac{z(\omega)z_{t,\omega}}{Z_t} \right)^{\theta-1} Y_t$$

$$F_s(\omega) = \int_{V_t(\omega)} Y_t \cdot v^\gamma.$$

Assume $u(C_s) = \ln(C_s)$, combine, and rearrange to get:

$$\frac{V_t(\omega)}{Y_t} = E \left[ \sum_{s=t+1}^{\infty} \beta^{s-t} \left( \frac{1}{\bar{\theta}} \left( \frac{z(\omega)}{Z_t} \right)^{\theta-1} \int_{V_t(\omega)} z_{t,\omega}^{\theta-1} v^\gamma \lambda(\omega) \right) \right].$$
Split up the integral and evaluate the first term, assuming w.l.g. that $\nu(\omega) \in \mathbb{N}$:

$$\int_{\nu(\omega)} z_{t,v}^{\theta-1} \lambda(dv) = \int_{v=1}^{\nu(\omega)} \epsilon_{s,[v]} \lambda(dv)$$

$$= \int_{1}^{2} \epsilon_{s,2} \lambda(dv) + \int_{2}^{3} \epsilon_{s,3} \lambda(dv) + \cdots + \int_{\nu(\omega)-1}^{\nu(\omega)} \epsilon_{s,\nu(\omega)} \lambda(dv)$$

$$= \sum_{n=1}^{\nu(\omega)-1} \epsilon_{s,n+1}$$

Now evaluate the second part of the integral that we split above:

$$\int_{\nu(\omega)} v^\gamma \lambda(dv) = \left( \frac{\nu(\omega)^{\gamma+1}}{1+\gamma} - \frac{1}{1+\gamma} \right).$$

Substituting back into the expression for firm value,

$$\frac{V_t(\omega)}{Z_t} = \sum_{s=t+1}^{\infty} \beta^s \left( \frac{z(\omega)^{\theta-1}}{\theta} \sum_{n=1}^{\infty} E \left[ \epsilon_{s,n+1} Z_{\theta,n}^{-1} \right] \right) - \sum_{s=t+1}^{\infty} \beta^s \left( \frac{\nu(\omega)^{\gamma+1}}{1+\gamma} - \frac{1}{1+\gamma} \right)$$

To a first-order approximation, the expectation is: $E \left[ \frac{\epsilon_{s,n+1}}{Z_{\theta,n}} \right] \approx \frac{\mu e}{\theta}$. Simplifying,

$$\frac{V_t(\omega)}{Y_t} \approx z(\omega)^{\frac{1+\gamma}{\gamma}(\theta-1)} \left( \frac{\mu e}{\theta} \frac{1+\gamma}{1+\gamma} \right) - z(\omega)^{\theta-1} \left( \frac{\mu e}{\theta} \right) + \left( \frac{1}{1+\gamma} \right)$$

Now combining with the covariance expression derived above:

$$\frac{\sigma_{s\omega}(\omega)}{V_t(\omega)} \approx \frac{\sigma^2}{Y_t} \cdot \frac{z(\omega)^{\theta-1} \left( \frac{\theta-1}{\gamma(\omega-(\theta-1))} \right) \left[ \frac{\mu e}{\theta} \frac{1+\gamma}{1+\gamma} \right]}{z(\omega)^{\frac{1+\gamma}{\gamma}(\theta-1)} \left( \frac{\mu e}{\theta} \frac{1+\gamma}{1+\gamma} \right) - z(\omega)^{\theta-1} \left( \frac{\mu e}{\theta} \right) + \frac{1}{1+\gamma}}.$$
Finally, using the expression for $z$, and collecting parameters to simplify,

$$
\frac{\sigma_{\omega}(\omega)}{V_t(\omega)} \approx \frac{1}{Y_t} \left( \frac{1}{\zeta_{v_1}} \left[ 1 - \left( \frac{z(\omega)}{z} \right)^{\zeta_{v_2}} \right] \zeta_{v_3} - \left( \frac{1}{z} \right)^{\zeta_{v_4}} \right),
$$

where

$$
\zeta_{v_1} := \frac{\theta - 1}{\gamma} \left( \frac{\sigma^2(\theta - 1)}{\theta - 1} \right), \quad \zeta_{v_2} := \frac{\theta - 1}{\gamma} \left[ \frac{\gamma[\kappa - (\theta - 1)] - (\theta - 1)}{\gamma} \right],
$$

$$
\zeta_{v_3} := \frac{1}{\gamma + 1}, \quad \zeta_{v_4} := (\theta - 1).
$$

Proposition 4.5 Let technology sets be those that firms choose in the non-stochastic steady state. Then firm $\omega$’s expected excess return is approximated to a second order by

$$
E \left[ r_t(\omega) - r_{f,t} \right] \approx \zeta_{v_1} \frac{\mu(\omega)}{V_t(\omega)} + \zeta_{v_2} \frac{\sigma_{\omega}(\omega)}{V_t(\omega)},
$$

where $I$ define firm $\omega$’s return as $r_t(\omega) = \left[ V_{t+1}(\omega) + \Pi_{t+1}(\omega) - F_{t+1}(\omega) \right] / V_t(\omega)$, and the risk-free rate as $r_{f,t} = m_{t+1}^{-1}$. Under parameter restrictions, expected excess returns decrease in firm productivity $z(\omega)$ for all $z(\omega)$ above a threshold.

Proof. Start with the definition of firm $\omega$’s stock return:

$$
r_{t+1}(\omega) = \frac{V_{t+1}(\omega) + \Pi_{t+1}(\omega) - F_{t+1}(\omega)}{V_t(\omega)}
= \frac{E \left[ \sum_{s=t+2}^{\infty} m_{t+1,s}(\Pi_s(\omega) - F_s(\omega)) \right] + \Pi_{t+1}(\omega) - F_{t+1}(\omega)}{V_t(\omega)}
= \frac{Y_{t+1} E \left[ \sum_{s=t+1}^{\infty} \beta^{s-(t+1)} \left( \frac{\Pi_s(\omega)}{Y_s} - \frac{F_s(\omega)}{Y_s} \right) \right]}{V_t(\omega)},
$$

where the third line assumes log utility and uses the definition of the household stochastic discount.
factor. Now take the time-$t$ conditional expectation:

\[
E_t [r_{t+1} (\omega)] = E_t \left[ \frac{Y_{t+1} E_{t+1} \left[ \sum_{s=t+1}^{\infty} \beta^{s-(t+1)} \left( \frac{\Pi_s (\omega)}{Y_s} - \frac{F_s (\omega)}{Y_s} \right) \right]}{V_t (\omega)} \right]
\]

\[
= E_t [Y_{t+1}] E_t \left[ \frac{E_{t+1} \left[ \sum_{s=t+1}^{\infty} \beta^{s-(t+1)} \left( \frac{\Pi_s (\omega)}{Y_s} - \frac{F_s (\omega)}{Y_s} \right) \right]}{V_t (\omega)} \right]
\]

\[
+ \text{Cov}_t \left( Y_{t+1}, E_{t+1} \left[ \sum_{s=t+1}^{\infty} \beta^{s-(t+1)} \left( \frac{\Pi_s (\omega)}{Y_s} - \frac{F_s (\omega)}{Y_s} \right) \right] \right)
\]

\[
\Rightarrow E_t [r_{t+1} (\omega) - r_{f,t+1}] = \frac{\text{Cov}_t \left( Y_{t+1}, E_{t+1} \left[ \sum_{s=t+1}^{\infty} \beta^{s-(t+1)} \left( \frac{\Pi_s (\omega)}{Y_s} - \frac{F_s (\omega)}{Y_s} \right) \right] \right)}{V_t (\omega)}.
\]

Consider the covariance term separately, and recall that zero serial correlation has been assumed for the random $\epsilon_{s,n}$'s:

\[
\text{Cov}_t \left( Y_{t+1}, E_{t+1} \left[ \sum_{s=t+1}^{\infty} \beta^{s-(t+1)} \left( \frac{\Pi_s (\omega)}{Y_s} - \frac{F_s (\omega)}{Y_s} \right) \right] \right)
\]

\[
= \text{Cov}_t \left( Y_{t+1}, \frac{\Pi_{t+1} (\omega)}{Y_{t+1}} - \frac{F_{t+1} (\omega)}{Y_{t+1}} \right)
\]

\[
= E_t [\Pi_{t+1} (\omega) - F_{t+1} (\omega)] - E_t [Y_{t+1}] E_t \left[ \frac{\Pi_{t+1} (\omega)}{Y_{t+1}} - \frac{F_{t+1} (\omega)}{Y_{t+1}} \right]
\]

\[
= E_t [\Pi_{t+1} (\omega)] - E_t [Y_{t+1}] E_t \left[ \frac{\Pi_{t+1} (\omega)}{Y_{t+1}} \right]
\]

\[
= E_t \left[ \frac{1}{V_t (\omega)} \int_{V_t (\omega)} \frac{z (\omega) z_{t+1,v}}{Z_{t+1}} \frac{\theta^{-1}}{\lambda (dv)} \right] - E_t [Y_{t+1}] E_t \left[ \int_{V_t (\omega)} \frac{1}{\theta} \left( \frac{z (\omega) z_{t+1,v}}{Z_{t+1}} \right)^{\theta-1} \lambda (dv) \right]
\]

\[
= E_t \left[ \frac{1}{\theta} Y_{t+1} \left( \frac{Z_{t+1} (\omega)}{Z_{t+1}} \right)^{\theta-1} \right] - E_t [Y_{t+1}] E_t \left[ \frac{1}{\theta} \left( \frac{Z_{t+1} (\omega)}{Z_{t+1}} \right)^{\theta-1} \right],
\]

where the second line uses the assumption of zero serial correlation. Now recall from (51) that
\[ Y_{t+1} = Z_{t+1}(K_{t+1})^{\alpha}(L)^{1-\alpha}, \text{ that } K_{t+1} \text{ is determined in } t, \text{ and that } L \text{ fixed, so} \]

\[
\text{Cov}_t \left( Y_{t+1}, E_{t+1} \left[ \sum_{s=t+1}^{\infty} \beta^{s-(t+1)} \left( \frac{\Pi_s(\omega)}{Y_s} - \frac{F_s(\omega)}{Y_s} \right) \right] \right) = \frac{(K_{t+1}^\alpha L^{1-\alpha})}{\theta} \left( E_t \left[ \frac{Z_{t+1}(\omega)^{\theta-1}}{Z_{t+1}^{\theta-2}} \right] - E_t [Z_{t+1}] E_t \left[ \left( \frac{Z_{t+1}(\omega)}{Z_{t+1}} \right)^{\theta-1} \right] \right)
\]

Next, second-order approximate the individual right-hand side expectations around the non-stochastic steady state values \( \mu(\omega) \) and \( \mu \). Starting with the first right-hand side expectation:

\[
E_t \left[ \frac{Z_{t+1}(\omega)^{\theta-1}}{Z_{t+1}^{\theta-2}} \right] \approx \frac{\mu(\omega)}{\theta} \frac{1}{2} \left( \mu^2 - 1 \right) + \frac{\mu(\omega)}{\theta} \left( \mu \right) \sigma^2 - \frac{1}{2} \left( \mu - 1 \right) \frac{\sigma^2}{\theta} \cdot \sigma_{\omega}(\omega).
\]

Now the second:

\[
E_t [Z_{t+1}] \approx \mu^\theta + \frac{1}{2} \left( \mu - 1 \right) \left( \mu - 1 \right) \mu^\theta \sigma^2.
\]

And the third:

\[
E_t \left[ \left( \frac{Z_{t}(\omega)}{Z_{\omega}} \right)^{\theta-1} \right] \approx \left( \frac{1}{\mu^2} + \sigma^2 \right) \mu(\omega) - \left( \frac{1}{2\mu^2} \right) \sigma_{\omega}(\omega).
\]

Substituting the approximations in the covariance expression, and rearranging,

\[
\text{Cov}_t \left( Y_{t+1}, E_{t+1} \left[ \sum_{s=t+1}^{\infty} \beta^{s-(t+1)} \left( \frac{\Pi_s(\omega)}{Y_s} - \frac{F_s(\omega)}{Y_s} \right) \right] \right) \approx \frac{(K_{t+1}^\alpha L^{1-\alpha})}{\theta} \left( \mu^2 - 1 \right) + \frac{(K_{t+1}^\alpha L^{1-\alpha})}{\theta} \left( \mu \right) \sigma^2 - \frac{1}{2} \left( \mu - 1 \right) \frac{\sigma^2}{\theta} \cdot \sigma_{\omega}(\omega)
\]

\[
\approx \zeta_1 \mu(\omega) + \zeta_2 \sigma_{\omega}(\omega),
\]

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where \( \zeta_{r1} \) and \( \zeta_{r2} \) are parameter collections given by:

\[
\zeta_{r1} := \left( \frac{K_{t+1}^\alpha L^{1-\alpha}}{\theta} \right) \left( \frac{\mu^{\frac{\theta-1}{\mu}}}{\theta} \right) \sigma^2 + \left( \frac{\mu^{\frac{\theta-1}{\mu}}}{\theta} \right) \left( \frac{1}{2} \right) \left( \frac{\theta-2}{\theta-1} \right) \left( \frac{1}{\theta-1} \right) \left( \frac{\sigma}{\mu} \right)^2 - 1 \right)
\]

\[
\zeta_{r2} := \left( \frac{K_{t+1}^\alpha L^{1-\alpha}}{\theta} \right) \left[ \left( \frac{1}{2} \right) \left( \frac{1}{\theta-1} \right) \left( \frac{1}{\mu^{\frac{\theta-1}{\mu}}} \right) \right] \left( 1 - \left( \frac{1}{2} \right) \left( \frac{\theta-2}{\theta-1} \right) \left( \frac{1}{\mu^{\frac{1}{\theta}}} \right) \right]
\]

and \( K_{t+1} \) is evaluated at its steady-state value. Finally, returning to the expression for expected excess returns,

\[
E_t [r_{t+1}(\omega) - r_{f,t+1}] \approx \zeta_{r1} \frac{\mu(\omega)}{V_t(\omega)} + \zeta_{r2} \frac{\sigma(\omega)}{V_t(\omega)}.
\]

\( \square \)

### A.1.3 Steady-state equilibrium

Equilibrium requires that the following market clearing conditions hold:

\[
c_{t,v}(\omega) = y_{t,v}(\omega)
\]

\[
L = \int_{\Omega} \int_{V(\omega)} l_{t,v}(\omega) \lambda(dvd\omega) + l_t
\]

\[
K_t = \int_{\Omega} \int_{V(\omega)} k_{t,v}(\omega) \lambda(dvd\omega) + k_t
\]

\[
\bar{I}_t = I_t + \int_{\Omega} \int_{V(\omega)} f_{s,v} \lambda(dvd\omega)
\]

\[
S_t(\omega) = 1.
\]

In the steady state equilibrium, the random productivities take their expected values (\( z_{t,v}^{\theta-1} = \mu_t, \forall v \in V \)), and capital and consumption are constant over time (\( C_{t+1} = C_t = C^*, K_{t+1} = K_t = K^* \)).

Under these conditions, solving for steady-state values of endogenous variables is straight-forward.

Begin by solving for the steady state wage and rental rate. In steady state, (39) becomes
1 = \beta(1 + r^* - \delta). Using (43) to substitute for \( r^* \):

\[
1 = \beta(1 + r^* - \delta) = \beta(1 - \delta + \alpha \mu \left( \frac{l^*}{k^*} \right)^{1-\alpha})
\]

\[
\Leftrightarrow \frac{k^*}{l^*} = \left[ \frac{\alpha \beta \mu}{1 - \beta(1 - \delta)} \right]^{\frac{1}{1-\alpha}}.
\]

Returning to (43) and evaluating at steady state,

\[
r^* = \alpha \mu \left( \frac{k^*}{l^*} \right)^{\alpha - 1} = \mu \left[ \frac{\alpha \beta \mu}{1 - \beta(1 - \delta)} \right]^{\alpha - 1} = 1 - \beta(1 - \delta) \beta.
\]

Now using (44),

\[
w^* = (1 - \alpha) \mu \left( \frac{k^*}{l^*} \right)^{\alpha} = (1 - \alpha) \mu \left[ \frac{\alpha \beta \mu}{1 - \beta(1 - \delta)} \right]^{\alpha-1}.
\]

Combining,

\[
\frac{r^*}{w^*} = \left( \frac{\alpha}{1 - \alpha} \right) \left[ \frac{1 - \beta(1 - \delta)}{\alpha \beta \mu} \right]^{\frac{1}{1-\alpha}}.
\]

Next, find an expression for the steady-state aggregate capital stock. Start with the definitions of aggregate capital and labor:

\[
K^* = \int_{\Omega} \int_{V(\omega)} k^*_v(\omega) \lambda(dvd\omega) + k^*
\]

\[
L = \int_{\Omega} \int_{V(\omega)} l^*_v(\omega) \lambda(dvd\omega) + l^*.
\]
Now use (36) and (45) to write

\[
L = \left(1 - \frac{\alpha}{\alpha}\right) \left(\frac{w^*}{r^*}\right) \left[\int_{\Omega} \int_{\nu(\omega)} k^*_\nu(\omega) \lambda(dv d\omega) + k^*\right]
\]

\[
\Leftrightarrow K^* = \left(\frac{\alpha}{1 - \alpha}\right) \left(\frac{r^*}{w^*}\right) L
\]

\[
= \left(\frac{\alpha}{1 - \alpha}\right)^2 \left[\frac{1 - \beta(1 - \delta)}{\alpha \beta \mu}\right] \frac{1 - \alpha}{\alpha} L.
\]

Recall that \(L\) is exogenous, so the above expression suffices. Next, use the law of motion for capital to find a steady-state expression for investment demand \(I_t\):

\[
K^* = I^* + (1 - \delta)K^*
\]

\[
\Leftrightarrow I^* = \delta K^*
\]

\[
= \delta \left(\frac{\alpha}{1 - \alpha}\right)^2 \left[\frac{1 - \beta(1 - \delta)}{\alpha \beta \mu}\right] \frac{1 - \alpha}{\alpha} L
\]
References


Forbes, K. J. (2012). The "big c": Identifying and mitigating contagion.


