In Defense of Information Leadership Share: A Response to Shrestha and Lee (2023)

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Abstract

Shrestha and Lee (2023) is a short note arguing that the information leadership share (ILS) measure by Putniņš (2013), building off of the information leadership (IL) measure by Yan and Zivot (2010), is flawed and not appropriate for measuring price leadership, and they argue for the continued use of information share (IS) and component share (CS) related measures instead of the ILS measure. In this paper, we argue that their critiques are flawed and their examples are misleading. Our analysis supports the usage of the ILS measure.

Keywords: information share, information leadership share, price discovery, structural moving average models.

JEL Classification: C32, G10

1 Introduction

Traditional price discovery measures include the information share (IS) measure of Hasbrouck (1995) and the component share (CS) measure of Gonzalo and Granger 1995. IS and CS metrics are based on estimates from a reduced-form vector error correction model (VECM). To provide structural interpretations of these measures, Yan and Zivot (2010) used a structural moving average (SMA) model to show that both IS and CS measures reflect the markets' responses to permanent shocks as well as responses to transitory noises given uncorrelated reduced form errors. Given the structural representations of IS and CS, they proposed the information leadership (IL) measure consisting of the ratio of IS to CS which they show to be free of transitory noise impacts given uncorrelated reduced-form errors. Putniņš (2013) further defined the information leadership share (ILS) measure, and used simulation evidence to show the ILS's robustness for leadership identification to differences in noise levels across markets.

Recently, Shrestha and Lee (2023) wrote a short note to argue that the ILS measure is flawed and is not appropriate for measuring price leadership and advocated the continued use of IS, CS, and the modified information share (MIS) measure by Lien and Shrestha (2009) and Lien and Shrestha (2014). Their main argument against the use of ILS is their Proposition 1, which is based on the SMA of Yan and Zivot (2010) under the assumption of uncorrelated reduced-form residuals, stating that equal IS and CS measures imply equal ILS measures. By considering a simple example model with identical IS and CS metrics and a counter example with a stylized dominant-satellite model, they present evidence supposedly supporting IS and CS against ILS. They also point out some inconsistencies in the simulation evidence in Putniņš (2013) based on their Proposition 1.

In this reply to their short note, we argue that the main critiques and examples in Shrestha and Lee (2023) against the IL/ILS measures are themselves flawed and misleading. In responding to their Proposition 1, we show that the equivalence of IS and CS metrics in the SMA model of Yan and Zivot (2010) with uncorrelated reduced-form residuals occurs only when both markets have the same instantaneous responses to the permanent shock. As a result, ILS correctly identifies equal price leadership while IS and CS may not. We restate their Proposition 1 to clarify this result and we reinterpret their simple example as supporting ILS over IS and CS.

We address the apparent inconsistency of the simulation evidence in Putniņš (2013) given Proposition 1 of Shrestha and Lee (2023). Our analysis shows that the simulation results in lower left corners of Tables 1-3 of Putniņš (2013) fall into one special scenario in which the second market's ratio of IS to CS is approaching 1 while the ratio of the first market is approaching zero. In the limit, we show that the ILS metric of the second market approaches the value of 1, instead of "0.5 or undefined" as claimed in Shrestha and Lee (2023).

We show that the analysis of price leadership using IS, CS, and ILS in the stylized dominant-satellite model is more subtle than the analysis presented in Shrestha and Lee (2023). The real issue, we argue as in Yan and Zivot (2007), is that IS, CS, and ILS are static measures of price discovery in that they only involve the contemporaneous impacts of permanent and transitory shocks. In this regard, the leading market is defined as the one with a larger instantaneous response (in absolute value) to the permanent innovation. However, as discussed in Yan and Zivot (2007), in a model with dynamics price leadership should be defined in terms of relative speeds of adjustment to the permanent shock and these relative speeds can be captured by the cumulative pricing errors associated with the permanent shock. From this perspective, a leading market should be the one with a smaller cumulative pricing error instead of the one with a larger instantaneous response to the permanent innovation. In this context, IS, CS, and ILS can give misleading results regarding price leadership. The counter example of Shrestha and Lee (2023) involves large negative responses to the permanent shock where the second market has a larger pricing error than the first market but also a larger instantaneous response to the permanent shock and so ILS mistakenly picks the second market as the price leader. Because IS and CS also respond to contemporaneous avoidance to transitory shocks, and in the counter example these noise responses are very large for market 2, IS and CS point to market 1 as the price leader but for the wrong reason. Only by examining the full dynamics of the pricing errors can we understand which market is really the price leader.

In short, our theoretical analysis and discussions on the definition of the leading market can help us better understand how and why a measure may succeed in price leadership identifications. We believe that our analysis can provide even stronger support for the usage of the IL/ILS measure in price discovery.

Our note is organized as follows. In Section 2, we briefly review the cointegration model upon which the common price discovery measures are derived. In Section 3, we examine the issues raised by Shrestha and Lee (2023) regarding the inappropriateness of using ILS as a measure of price discovery. We provide a structural representation of the MIS measure in Section 4 and show that is has the same limitations as IS. We provide concluding remarks in Section 5.

2 Information Share Measures

2.1 IS and CS based on VECM

Let $\mathbf{p}_t = (p_{1t}, p_{2t})'$ denote a vector of log prices for two assets that are intergated of order 1, or I(1), and assume that $\mathbf{p}_t$ is cointegrated with cointegrating vector $\boldsymbol{\beta} = (1, -1)'$. Price discovery measures are typically derived from a reduced-form VECM of the form:

$$\Delta \mathbf{p}_t = \boldsymbol{\alpha}\boldsymbol{\beta}' \mathbf{p}_{t-1} + \sum_{j=1}^{k} \boldsymbol{\Gamma}_j \Delta \mathbf{p}_{t-j} + \boldsymbol{\varepsilon}_t, \tag{1}$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)'$ is the error correction vector, $\boldsymbol{\Gamma}_j$ $(i = 1, \ldots, k)$ are the short-run coefficient matrices, and $\boldsymbol{\varepsilon}_t = (\varepsilon_{1t}, \varepsilon_{2t})'$ is the vector of reduced-form residuals with $E[\boldsymbol{\varepsilon}_t] = \mathbf{0}$ and $E[\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t'] = \boldsymbol{\Omega}$.

Hasbrouck (1995) transforms the VECM into a reduced-form vector moving average

(VMA) model:

$$\Delta \mathbf{p}_t = \boldsymbol{\Psi}(L)\boldsymbol{\varepsilon}_t = \boldsymbol{\varepsilon}_t + \boldsymbol{\Psi}_1\boldsymbol{\varepsilon}_{t-1} + \boldsymbol{\Psi}_2\boldsymbol{\varepsilon}_{t-2} + \cdots , \tag{2}$$

and its integrated form (or Beveridge-Nelson decomposition):

$$\mathbf{p}_t = \mathbf{p}_0 + \boldsymbol{\Psi}(1)\sum_{s=1}^{t}\boldsymbol{\varepsilon}_s + \boldsymbol{\Psi}^*(L)\boldsymbol{\varepsilon}_t, \tag{3}$$

where $\boldsymbol{\Psi}(1) = \sum_{k=0}^{\infty}\boldsymbol{\Psi}_k$ with $\boldsymbol{\Psi}(L)$ and $\boldsymbol{\Psi}^*(L)$ being matrix polynomials in the lag operator, L, and $\boldsymbol{\Psi}^*(k) = -\sum_{j=k+1}^{\infty}\boldsymbol{\Psi}_j$.

The matrix $\boldsymbol{\Psi}(1)$ contains the cumulative impacts of the innovation $\boldsymbol{\varepsilon}_t$ on all future price movements, and acts as a measure of the long-run impact of $\boldsymbol{\varepsilon}_t$ on prices. As shown in Hasbrouck (1995), the rows of $\boldsymbol{\Psi}(1)$ are identical given $\boldsymbol{\beta} = (1, -1)'$. Denote $\boldsymbol{\psi} = (\psi_1, \psi_2)'$ as the common row vector of $\boldsymbol{\Psi}(1)$, and define the permanent innovation as:

$$\eta_t^P = \boldsymbol{\psi}'\boldsymbol{\varepsilon}_t = \psi_1\varepsilon_{1t} + \psi_2\varepsilon_{2t}. \tag{4}$$

This common efficient price $m_t = m_{t-1} + \eta_t^P$ evolves as a random walk driven by the permanent shock η_t^P.

The IS measure of Hasbrouck (1995) quantifies each price series' contribution to price discovery based on the share of the variance of the efficient price that is attributable to this series. Under the case of a diagonal covariance matrix $\boldsymbol{\Omega}$, the IS measures for each market are uniquely defined as:

$$\mathrm{IS}_1 = \frac{\psi_1^2\sigma_1^2}{\boldsymbol{\psi}'\boldsymbol{\Omega}\boldsymbol{\psi}}, \quad \mathrm{IS}_2 = \frac{\psi_2^2\sigma_2^2}{\boldsymbol{\psi}'\boldsymbol{\Omega}\boldsymbol{\psi}}. \tag{5}$$

The IS measures take a positive value between 0 and 1 by construction.

The CS measure by Booth et al. (1999), Chu et al. (1999), and Harris et al. (2002)

quantifies each market's contribution to the common efficient price component by their weights:

$$\mathrm{CS}_1 = \frac{\psi_1}{\psi_1 + \psi_2}, \quad \mathrm{CS}_2 = \frac{\psi_2}{\psi_1 + \psi_2}. \tag{6}$$

By construction the CS values sum to one. The above expression also indicates that IS is a variance-weighted version of CS when the reduced-form innovations are uncorrelated as noted in Yan and Zivot (2010).

2.2 SMA and ILS

As noted in Yan and Zivot (2010) and Lehmann (2002), it is impossible to get a clear structural interpretation of IS and CS measures since they are based on residuals from a reduced-form VECM. Yan and Zivot (2010) used a structural cointegration model with independent permanent and transitory shocks to derive structural representations of IS and CS.

The structural moving average (SMA) representation of $\Delta\mathbf{p}_t$ in Yan and Zivot (2010) is given as:

$$\Delta\mathbf{p}_t = \mathbf{D}(L)\boldsymbol{\eta}_t = \mathbf{D}_0\boldsymbol{\eta}_t + \mathbf{D}_1\boldsymbol{\eta}_{t-1} + \mathbf{D}_2\boldsymbol{\eta}_{t-2} + \dots \tag{7}$$

where the elements of $\{\mathbf{D}_k\}_{k=0}^{\infty}$ are 1-summable, $\mathbf{D}(L) = \sum_{k=0}^{\infty}\mathbf{D}_k L^k$, and $\mathbf{D}_0$ is invertible. The innovation to the common efficient price of the asset, η_t^P, is labeled the permanent shock and the noise innovation, η_t^T, is labeled the transitory shock so that $\boldsymbol{\eta}_t = (\eta_t^P, \eta_t^T)'$. These structural shocks are assumed to be serially and mutually uncorrelated with diagonal covariance matrix $\mathbf{C} = \mathrm{diag}(\sigma_P^2, \sigma_T^2)$. The matrix $\mathbf{D}_0$ contains the initial impacts of the structural shocks on $\Delta\mathbf{p}_t$, and defines the contemporaneous correlation structure of $\Delta\mathbf{p}_t$:

$$\mathbf{D}_0 = \begin{bmatrix} d_{0,1}^P & d_{0,1}^T \\ d_{0,2}^P & d_{0,2}^T \end{bmatrix}. \tag{8}$$

The defining characteristic of η_t^T is that it is uncorrelated with η_t^P, and has no long-run effect on price levels. Hence, the long-run impact matrix $\mathbf{D}(1)$ of the structural innovations $\boldsymbol{\eta}_t$ takes the form

$$\mathbf{D}(1) = \begin{bmatrix} d_1^P(1) & d_1^T(1) \\ d_2^P(1) & d_2^T(1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}. \tag{9}$$

Given this set-up, Yan and Zivot (2010) derived the structural representations of CS as:

$$\mathrm{CS}_1 = \frac{d_{0,2}^T}{d_{0,2}^T - d_{0,1}^T}, \quad \mathrm{CS}_2 = -\frac{d_{0,1}^T}{d_{0,2}^T - d_{0,1}^T}. \tag{10}$$

The structural representations of CS only involve the transitory structural parameters $d_{0,i}^T$ $(i = 1, 2)$. Instead of measuring the relative strength of how a given market price responds to new information, CS quantifies the other market's relative response to contemporaneous transitory frictions.

To derive the structural representation for IS, Yan and Zivot (2010) consider the special case when the reduced-form innovations $\boldsymbol{\varepsilon}_t$ are uncorrelated.[1] Under this case, attribution of the reduced-form covariance to individual markets is irrelevant and the IS measures are unique. Yan and Zivot (2010) show that $\mathrm{cov}(\varepsilon_{1t}, \varepsilon_{2t}) = 0$ occurs when:

$$\frac{\sigma_T^2}{\sigma_P^2} = \frac{d_{0,1}^P d_{0,2}^P}{-d_{0,1}^T d_{0,2}^T}, \tag{11}$$

and all elements of $\mathbf{D}_0$ are non-zero (hence $|\mathbf{D}_0| \neq 0$). Under this special case, the IS

[1] Shen et al. (2024) consider the general case of correlated reduced-form residuals.

measures can be uniquely defined in terms of the structural parameters:

$$\mathrm{IS}_1 = \frac{d^P_{0,1} d^T_{0,2}}{d^P_{0,1} d^T_{0,2} - d^T_{0,1} d^P_{0,2}}, \quad \mathrm{IS}_2 = \frac{-d^T_{0,1} d^P_{0,2}}{d^P_{0,1} d^T_{0,2} - d^T_{0,1} d^P_{0,2}}. \tag{12}$$

The structural representations of IS consist of contemporaneous responses to both permanent and transitory shocks. Compared to CS, Yan and Zivot (2010) argue that IS is more appropriate for measuring price discovery because it contains individual market's responses to permanent shocks or new information, $d^P_{0,i}$ $(i = 1, 2)$, whereas CS does not.

In the special case of uncorrelated reduced-form residuals, Yan and Zivot (2010) quantify the relative contemporaneous impact of the permanent shocks by the information leadership (IL) metric:

$$\mathrm{IL}_1 = \left| \frac{\mathrm{IS}_1/\mathrm{CS}_1}{\mathrm{IS}_2/\mathrm{CS}_2} \right| = \left| \frac{d^P_{0,1}}{d^P_{0,2}} \right|, \quad \mathrm{IL}_2 = \left| \frac{\mathrm{IS}_2/\mathrm{CS}_2}{\mathrm{IS}_1/\mathrm{CS}_1} \right| = \left| \frac{d^P_{0,2}}{d^P_{0,1}} \right|. \tag{13}$$

The IL measure only depends on each market's initial responses to the permanent shock and provide a straightforward and intuitive measure of price discovery.

To make IL easier to interpret and more comparable to IS and CS, Putniņš (2013) defined the information leadership shares (ILS):

$$\mathrm{ILS}_1 = \frac{\mathrm{IL}_1}{\mathrm{IL}_1 + \mathrm{IL}_2}, \quad \mathrm{ILS}_2 = \frac{\mathrm{IL}_2}{\mathrm{IL}_1 + \mathrm{IL}_2}. \tag{14}$$

The ILS measures lie within the unit interval by construction, and a value above (below) 0.5 indicates the price series leads (does not lead) the adjustment process to new information.

3 Issues with the Analysis in Shrestha and Lee (2023)

3.1 Propostion 1 of Shrestha and Lee (2023)

After reviewing the IS, CS, and ILS measures, Shrestha and Lee (2023) establish the following proposition (page 5, Section 2.1):

Proposition 1: If $\text{IS}_i = \text{CS}_i$, $i = 1, 2$, $\text{IS}_i \neq 0$, and $\text{CS}_i \neq 1$, then $\text{ILS}_1 = \text{ILS}_2 = 0.5$.

The above proposition suggests that if IS and CS agree with each other, then the ILS of the two markets will be equal regardless of the values of IS and CS. At first glance, this proposition seems to provide evidence against the usefulness of ILS as a price leadership measure. However, a further examination of the case when $\text{IS}_i = \text{CS}_i$ with uncorrelated reduced form residuals reveals the following for $i = 1$:[2]

$$\underbrace{\frac{d_{0,1}^P d_{0,2}^T}{d_{0,1}^P d_{0,2}^T - d_{0,1}^T d_{0,2}^P}}_{IS_1} = \underbrace{\frac{d_{0,2}^T}{d_{0,2}^T - d_{0,1}^T}}_{CS_1} \quad \Leftrightarrow \quad d_{0,1}^P = d_{0,2}^P \tag{15}$$

when $d_{0,i}^T \neq 0$ for $i = 1, 2$ and $d_{0,2}^T \neq d_{0,1}^T$. The above result shows that $\text{IS}_i = \text{CS}_i$ occurs only when the initial responses $d_{0,1}^P$ and $d_{0,2}^P$ to the permanent shock η_t^P are equal and, as a result, $\text{ILS}_i = 0.5$. So, we see that the ILS measure correctly identifies an equal leadership when both markets' initial responses to a permanent shock are the same.

Hence, a close examination of Proposition 1 of Shrestha and Lee (2023) actually provides support for the ILS measure. We summarize the above findings in the following modified proposition:

Proposition 1^***:** In the SMA model of Yan and Zivot (2010) with uncorrelated reduced-form residuals, when $d_{0,1}^P = d_{0,2}^P$, $d_{0,i}^P \neq 0$ and $d_{0,i}^T \neq 0$ for $i = 1, 2$ and $d_{0,2}^T \neq d_{0,1}^T$, then $\text{IS}_i = \text{CS}_i$, $i = 1, 2$, and $\text{ILS}_1 = \text{ILS}_2 = 0.5$.

[2] An analogous result holds for $i = 2$.

Our Proposition 1* clarifies the fact that IS and CS measures coincide only when price leadership is distributed evenly across the two markets. With equal leadership, IS and CS may take values other than 0.5, and hence give misleading leadership results even though these two measures may take the same value. So, the IS and CS measures alone are not sufficient for identifying the price leader. In comparison, the IL and ILS measures, which combine IS and CS, can always identify equal leadership under this case. By taking ratios of these IS and CS measures, parameters involving transitory responses ($d^T_{0,i}$) cancel out and the IL and ILS metrics only depend on the permanent responses ($d^P_{0,i}$) as shown in Eq. (13).

The above modified proposition above only provides support for the ILS measure under assumption of uncorrelated reduced-form residuals. For the more generalized situation with correlated residuals, it is possible that ILS may fail to identify the correct price leader. Shen et al. (2024) provides a simple modification of the ILS measure to make it valid for the case of correlated reduced-form residuals.

3.2 An example when IS is equal to CS

Our Proposition 1* provides support for the ILS measure in the simple example in Section 3.1 of Shrestha and Lee (2023) where IS = CS. In this example, $\text{IS}_1 = \text{CS}_1 = 0.99$ and $\text{IS}_2 = \text{CS}_2 = 0.01$ implies that $d^P_{0,1} = d^P_{0,2}$ and $d^T_{0,2} = -99d^T_{0,1}$. As we can see, the IS and CS measures incorrectly identify the first market as the price leader even though both markets respond equally to the permanent shock. In contrast, $\text{ILS}_1 = \text{ILS}_2 = 0.5$ correctly indicates that both market contribute equally to the price discovery process.

3.3 Simulation Analysis by Putniņš (2013)

To refute the simulation evidence in Putniņš (2013) in favor of ILS, Shrestha and Lee (2023) claim that "the value of ILS_2 should either be close to 0.5 or undefined" when the values of CS_2 and IS_2 are close to 1 as in the bottom left part of the Tables 1-2 of Putniņš (2013) (e.g. $\delta_2 = 2, 1, 0$ and $\sigma_{s_2} = 0, 1$). But as shown in Table 3 of Putniņš (2013), the mean values of ILS_2 are more than 0.97, contradicting Proposition 1 of Shrestha and Lee (2023) and the arguments in Section 3.3 of Shrestha and Lee (2023) indicating an undefined ILS.

However, their criticisms on the inconsistent values of ILS are invalid. Firstly, the bottom left parts of Tables 1-2 of Putniņš (2013) do not necessarily imply that $\text{IS}_2 = \text{CS}_2$. Even though mean values of CS_2 and IS_2 are very close to 1 with small standard deviations, these two measures may differ and may take values other than 1 as long as the standard deviations are not zero. Hence, it is not necessaryily true that ILS should be close to 0.5.

Secondly, notice that the ILS measures can be re-written as:

$$\text{ILS}_1 = \frac{(\text{IS}_1/\text{CS}_1)^2}{(\text{IS}_1/\text{CS}_1)^2 + (\text{IS}_2/\text{CS}_2)^2}, \quad \text{ILS}_2 = \frac{(\text{IS}_2/\text{CS}_2)^2}{(\text{IS}_1/\text{CS}_1)^2 + (\text{IS}_2/\text{CS}_2)^2}. \tag{16}$$

The above ILS measure can be well defined as long as none of these metrics are exactly 0 or 1. For the extreme case when $\text{IS}_2 \to 1$ and $\text{CS}_2 \to 1$, we have $\text{IS}_2/\text{CS}_2 \to 1$ and the limit of IS_1/CS_1 can be shown as:

$$\frac{\text{IS}_1}{\text{CS}_1} \to \begin{cases} 0, & \text{if } \text{IS}_1 < \text{CS}_1 \\ 0.5, & \text{if } \text{IS}_1 = \text{CS}_1 \\ \infty, & \text{if } \text{IS}_1 > \text{CS}_1. \end{cases} \tag{17}$$

Given the representation of ILS_i in Eq. (16) and the above limits of IS_1/CS_1, we have the following:

$$\text{ILS}_1 \to 0, \quad \text{ILS}_2 \to 1 \quad \text{if} \quad \text{IS}_1/\text{CS}_1 \to 0; \tag{18}$$

$$\text{ILS}_1 \to 0.5, \quad \text{ILS}_2 \to 0.5 \quad \text{if} \quad \text{IS}_1/\text{CS}_1 \to 1. \tag{19}$$

$$\text{ILS}_1 \to 1, \quad \text{ILS}_2 \to 0 \quad \text{if} \quad \text{IS}_1/\text{CS}_1 \to \infty, \tag{20}$$

As the above results indicate, depending on the relative magnitude of IS_1 to CS_1, ILS_2 can take a value of 0, 0.5, or 1, not “0.5 or undefined” as stated in Shrestha and Lee (2023). The simulation evidence in the bottom left part of Table 3 in Putniņš (2013) with ILS_2 close to one just implies that the first scenario in Eq. (18) is more plausible in their simulation.

3.4 The Dominant-Satellite Model

The counter example considered in Section 3.2 of Shrestha and Lee (2023) builds off the dominant-satellite model of Yan and Zivot (2010):

$$p_{1t} = m_t + s_{1t}, \quad p_{2t} = m_{t-1} + s_{2t}, \quad m_t = m_{t-1} + \eta_t^P, \tag{21}$$

$$s_{it} = b_{0,i}^P \eta_t^P + b_{0,i}^T \delta \eta_t^T, \quad i = 1, 2 \tag{22}$$

with $b_{01}^P = b_{02}^P = -2$, $b_{01}^T = -\delta$, and $b_{02}^T = w\delta$. In the above model, a permanent shock to the common efficient price m_t will be spontaneously incorporated into Market 1 with a tracking error s_{1t}, which reacts to both η_t^P and η_t^T, while Market 2 tracks the efficient price with a lag but has a tracking error $(s_{2t} - \eta_t^P)$.[3]

As shown in Yan and Zivot (2010), the above model implies that

$$\mathbf{D}_0 = \begin{bmatrix} 1 + b_{01}^P & b_{01}^T \\ b_{02}^P & b_{02}^T \end{bmatrix} = \begin{bmatrix} -1 & -\delta \\ -2 & \delta w \end{bmatrix}, \tag{23}$$

$$\mathbf{D}_1 = \begin{bmatrix} -b_{01}^P & -b_{01}^T \\ 1 - b_{02}^P & -b_{02}^T \end{bmatrix} = \begin{bmatrix} 2 & \delta \\ 3 & -\delta w \end{bmatrix}, \tag{24}$$

and $\mathbf{D}_k = \mathbf{0}$ for $k \geq 2$. Moreover, the long-run impact matrix of the structural innovations

[3] Note that tracking errors can be denoted as:

$$p_{1t} - m_t = s_{1t} = b_{0,1}^P \eta_t^P + b_{0,1}^T \eta_t^T,$$
$$p_{2t} - m_t = s_{2t} - \eta_t^P = (b_{0,2}^P - 1)\eta_t^P + b_{0,2}^T \eta_t^T.$$

When $b_{01}^P = b_{02}^P = -2$, $b_{01}^T = -\delta$, and $b_{02}^T = w\delta$, we can see the variance of Market 2's tracking error is larger than that of Market 1 when $w > 1$.

has the form

$$\mathbf{D}(1) = \mathbf{D}_0 + \mathbf{D}_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \tag{25}$$

which implies that both prices will increase by one after one period given a one-unit shock to η_t^P.

We depict the responses of prices of these two markets to a unit of the permanent shock in Figure 1. As Figure 1 shows, at time 0, the price of Market 1 first drops by -1, and then adjusts to its permanent level one period later. For Market 2, its price first drops to -2, a level further away from its permanent level than Market 1, and then adjusts to the permanent level at time 1.

[Insert Figure 1 about here.]

This is an unusual example because the initial responses to the permanent shock are large and in the opposite direction to what is typically expected.

If we define the pricing error as the difference between the price response and its permanent level, then we see that Market 1 has a smaller pricing error than Market 2 at time 0. In the terminology of Yan and Zivot (2010), Market 1 and Market 2 have the same adjustment speed toward their long-run equilibrium, but Market 1 has a smaller accumulative pricing error. In this sense, we can say that Market 1 is leading the price discovery process between these two markets, just as the DGP in Eq. (21) indicates.

Let's next examine what price discovery measures can tell us about the leadership between these two markets. By choosing $\sigma_P^2 = \frac{\sigma_T^2 w \delta^2}{2}$ with $w > 0$, the reduced-form innovations are uncorrelated. Then, the price discovery measures are shown by Shrestha and Lee (2023) as:

$$
\begin{aligned}
\mathrm{IS}_1 &= \frac{d^P_{0,1} d^T_{0,2}}{d^P_{0,1} d^T_{0,2} - d^T_{0,1} d^P_{0,2}} = \frac{w}{2+w}, & \mathrm{IS}_2 &= \frac{-d^T_{0,1} d^P_{0,2}}{d^P_{0,1} d^T_{0,2} - d^T_{0,1} d^P_{0,2}} = \frac{2}{2+w}, \\
\mathrm{CS}_1 &= \frac{d^T_{0,2}}{d^T_{0,2} - d^T_{0,1}} = \frac{w}{1+w}, & \mathrm{CS}_2 &= -\frac{d^T_{0,1}}{d^T_{0,2} - d^T_{0,1}} = \frac{1}{1+w}, \\
\mathrm{IL}_1 &= \left| \frac{d^P_{0,1}}{d^P_{0,2}} \right| = 0.5, & \mathrm{IL}_2 &= \left| \frac{d^P_{0,2}}{d^P_{0,1}} \right| = 2, \\
\mathrm{ILS}_1 &= \frac{\mathrm{IL}_1}{\mathrm{IL}_1 + \mathrm{IL}_2} = 0.2, & \mathrm{ILS}_2 &= \frac{\mathrm{IL}_2}{\mathrm{IL}_1 + \mathrm{IL}_2} = 0.8.
\end{aligned}
$$

The above results indicate that ILS_1 is 0.2 regardless of the values of w and δ, indicating the first market to be a satellite market instead of a leading market. In comparison, when $w > 2$, IS_1 is larger than 0.5, identifying the first market to be a dominant market. And when $w > 1$, CS_1 is larger than 0.5, implying that the first market is a dominant market. The counter example in Shrestha and Lee (2023) considers a special case with $w = 10$. Under this case, $\mathrm{IS}_1 = 0.833$ and $\mathrm{CS}_1 = 0.909$, which correctly identifies the leading role of the first market. However, this result occurs for the wrong reason. IS and CS identify the first market as the leader because the second market has a very large contemporaneous response to the transitory shock and not because the first market has a larger response to the permanent shock.

The above counter example seems to provide support for IS and CS against ILS. However, IS and CS can also give misleading leadership results when $w < 1$. The issue here is not just about choosing between these three measures. The issue is how we should define a leading market when there is non-trivial dynamics.

According to the analysis above, IL and ILS measures always have exact structural representations involving instant responses to the permanent innovation when reduced-form errors are uncorrelated, while IS and CS measures both involve instantaneous responses to the permanent innovation as well as instantaneous responses to the transitory noise. In this sense, ILS is a more trustworthy measure of leadership if we are defining the leader as the market with an instantaneous response to the permanent shock that is typical in size and direction.

The failure of ILS in this example stems from the typical definition of a market leader. Conventionally, the market with a larger absolute value of the initial response to the permanent shock ($|d^P_{0,i}|$) is defined as the leader. Under this definition, which we call the *instantaneous response rule*, the market with $\mathrm{IL}_i > 1$ and $\mathrm{ILS}_i > 0.5$ is chosen as the leading market. However, as the current example shows this rule is not always reliable.

The instantaneous response rule may not be the best way to define a price leader in a model with dynamics because the absolute magnitudes of the instantaneous responses to the permanent shock may not accurately reflect the price discovery dynamics. To illustrate, in the current example both prices will eventually increase by one unit given a one-unit shock to η^P_t. So, if the eventual response to a permanent shock is unity, another way to define the leading market is the market with the smaller instantaneous absolute pricing error $|d^P_{0,i} - 1|$, instead of the one with a larger instantaneous absolute response $|d^P_{0,i}|$. We call this the *instantaneous pricing error rule.*[4]

The instantaneous pricing error rule is related to how Yan and Zivot (2007) define price discovery in a dynamic model. They argue that a market's contribution to price discovery should be measured by the relative speed to which its observed price moves to the new fundamental value following a shock to the efficient price and this can be captured by the magnitudes of each market's cumulative pricing errors in the adjustment to the new fundamental value.[5]

In the above counter example, $d^P_{0,1} = -1$ and $d^P_{0,2} = -2$, both of which are negative with the latter taking a larger absolute value. The second market is chosen as a leading market by ILS under the instantaneous response rule. However, since $|d^P_{0,1} - 1| < |d^P_{0,2} - 1|$ the first

[4]Applying the instantaneous pricing rule requires the separate identification of $d^P_{0,1}$ and $d^P_{0,2}$. This is not possible using IS, CS, or ILS.

[5]Yan and Zivot (2007) focused on the dynamics of the price discovery process. In this short note, we stay with the static point of view of price discovery. However, our analysis and discussion can be easily extended to the dynamic process with a leader defined as the one with a smaller cumulative pricing error instead of a smaller instant pricing error.

market is determined as leading market under the instantaneous pricing error rule. In a model with dynamics, the problem with ILS is that the market with a larger initial response to the permanent innovation may be associated with a larger instantaneous pricing error. In another words, the leader defined under the instantaneous response rule may not be the leader under the instantaneous pricing error rule.

Usually, the instantaneous responses $d_{0,i}^P$ take values between 0 and 1 or nearly so. Then the market with a larger $|d_{0,i}^P|$ will also have a smaller pricing error ($|d_{0,1}^P - 1|$) and ILS will correctly identify the leader (while IS and CS may not). However, when a case involves negative values of $d_{0,i}^P$ (or values much larger than 1), the instantaneous pricing error rule will more reliably identify the correct leader while the instantaneous response rule (and hence ILS) may not.

In the dominant-satellite example, we depict the leading market in a diagram of $(d_{0,1}^P, d_{0,2}^P)$ under the instantaneous response rule and the instantaneous pricing error rule in Figures 2 and 3, respectively.

[Insert Figure 2 about here.]

[Insert Figure 3 about here.]

When $|d_{0,1}^P| > |d_{0,2}^P|$, market 1 is defined as the market leader using the instantaneous response rule only for the shaded green area in Figure 2. When $|d_{0,1}^P - 1| > |d_{0,2}^P - 1|$, market 2 is defined as the market leader using the instantaneous pricing error rult only for the shaded orange area in Figure 3 depicts. Putting this two diagrams in one graph, we can see that these two rules agree with each other on the market leader in the shaded area $0 < d_{0,1}^P + d_{0,2}^P < 2$ in Figure 4 depicts. As long as $0 < d_{0,1}^P + d_{0,2}^P < 2$, the market with a larger $|d_{0,i}^P|$ will also have a smaller pricing error ($|d_{0,i}^P - 1|$). And as a result, the IL and ILS measures will correctly identify the leading market. Otherwise, the IL and ILS measures will incorrectly identify the leading market.

[Insert Figure 4 about here.]

In many empirical settings, it is unusual to have very large negative (undershooting) or very large positive (overshooting) instantaneous responses to new information and so the condition $0 < d^P_{0,1} + d^P_{0,2} < 2$ is likely satisfied.

4 Issues with Modified Information Share

In their conclusion, Shrestha and Lee (2023) advocate for the use the modified IS (MIS) measure of Lien and Shrestha (2009) and Lien and Shrestha (2014) to eliminate the bound issue of IS. Here, we show that the MIS measure, like the IS measure, incorporates transitory responses and could yield misleading results regarding price discovery.

The MIS is defined by Lien and Shrestha (2009) as:

$$\text{MIS}_i = \frac{([\boldsymbol{\psi}'\mathbf{F}^*]_i)^2}{\boldsymbol{\psi}'\boldsymbol{\Omega}\boldsymbol{\psi}} = \frac{(\boldsymbol{\psi}^*_i)^2}{\sum_{i=1}^n (\boldsymbol{\psi}^*_i)^2}, \tag{26}$$

where $\boldsymbol{\psi}^{*\prime} = \boldsymbol{\psi}'\mathbf{F}^*$, $\mathbf{F}^* = [\mathbf{G}\boldsymbol{\Lambda}^{-1/2}\mathbf{G}'\mathbf{V}^{-1}]^{-1}$ with $\mathbf{G}$ being a matrix with eigenvectors of the correlation matrix of the reduced-form residuals as columns and $\boldsymbol{\Lambda}$ representing the diagonal matrix with the corresponding eigenvalues as diagonal elements.

Consider the bivariate case on page 385 of Lien and Shrestha (2009) where

$$\mathbf{G} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, \quad \boldsymbol{\Lambda} = \begin{pmatrix} 1+\rho & 0 \\ 0 & 1+\rho \end{pmatrix},$$

and

$$\mathbf{F}^* = \begin{pmatrix} 0.5(\sqrt{1+\rho} + \sqrt{1-\rho})\sigma_1 & 0.5(\sqrt{1+\rho} - \sqrt{1-\rho})\sigma_1 \\ 0.5(\sqrt{1+\rho} - \sqrt{1-\rho})\sigma_2 & 0.5(\sqrt{1+\rho} + \sqrt{1-\rho})\sigma_2 \end{pmatrix}.$$

Substituting relevant terms into the MIS formula in Eq. (26), we get the following bivariate formula for MIS:

$$\text{MIS}_i = \frac{\psi_i^2\sigma_i^2(1+\sqrt{1-\rho^2})/2 + \psi_j^2\sigma_j^2(1-\sqrt{1-\rho^2})/2 + \psi_i\psi_j\sigma_{i,j}}{\boldsymbol{\psi}'\boldsymbol{\Omega}\boldsymbol{\psi}}, \tag{27}$$

for $i = 1, 2$. As the above expression shows, the MIS measure decomposes the variance contribution to each market more equally than the IS measure does and coincides with the IS measure when $\rho = 0$.

To provide a structural representation for the MIS measure, we can substitute $\psi_1 = \frac{d_{0,2}^T}{|D_0|}$, $\psi_2 = \frac{-d_{0,1}^T}{|D_0|}$ and relevant terms from the structural representation of the reduced-form residual covariance matrix into the above expression to get:[6]

$$\begin{aligned}
\text{MIS}_1 &= \frac{1}{2} + \frac{1}{2}\frac{d_{0,1}^P d_{0,2}^T + d_{0,1}^T d_{0,2}^P}{d_{0,1}^P d_{0,2}^T - d_{0,1}^T d_{0,2}^P}\sqrt{1-\rho^2}, \\
\text{MIS}_2 &= \frac{1}{2} - \frac{1}{2}\frac{d_{0,1}^P d_{0,2}^T + d_{0,1}^T d_{0,2}^P}{d_{0,1}^P d_{0,2}^T - d_{0,1}^T d_{0,2}^P}\sqrt{1-\rho^2}.
\end{aligned} \tag{28}$$

As we can see, the structural representation of MIS is a complex combination of contemporaneous responses to both permanent and transitory shocks. As a result, it is possible that MIS identifies one market as the leader even though that market may have a smaller permanent response (i.e., smaller $|d_{0,i}^P|$) or a larger pricing error (i.e., larger $|d_{0,i}^P - 1|$) .

5 Conclusion

In this short note, we provide responses to the criticisms from Shrestha and Lee (2023) regarding the use of the ILS measure of Putniņš (2013) as a price discovery metric. We argue that the proposition and counter examples in Shrestha and Lee (2023) are flawed

[6] In the SMA model of Yan and Zivot (2010), the variance-covariance matrix of the reduced-form errors takes a form:

$$\boldsymbol{\Omega} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} = \begin{pmatrix} (d_{0,1}^P)^2\sigma_P^2 + (d_{0,1}^T)^2\sigma_T^2 & d_{0,1}^P d_{0,2}^P\sigma_P^2 + d_{0,1}^T d_{0,2}^T\sigma_T^2 \\ d_{0,1}^P d_{0,2}^P\sigma_P^2 + d_{0,1}^T d_{0,2}^T\sigma_T^2 & (d_{0,2}^P)^2\sigma_P^2 + (d_{0,2}^T)^2\sigma_T^2 \end{pmatrix}.$$

and misleading and, counter to their claims, are actually supportive for the use of the ILS measure.

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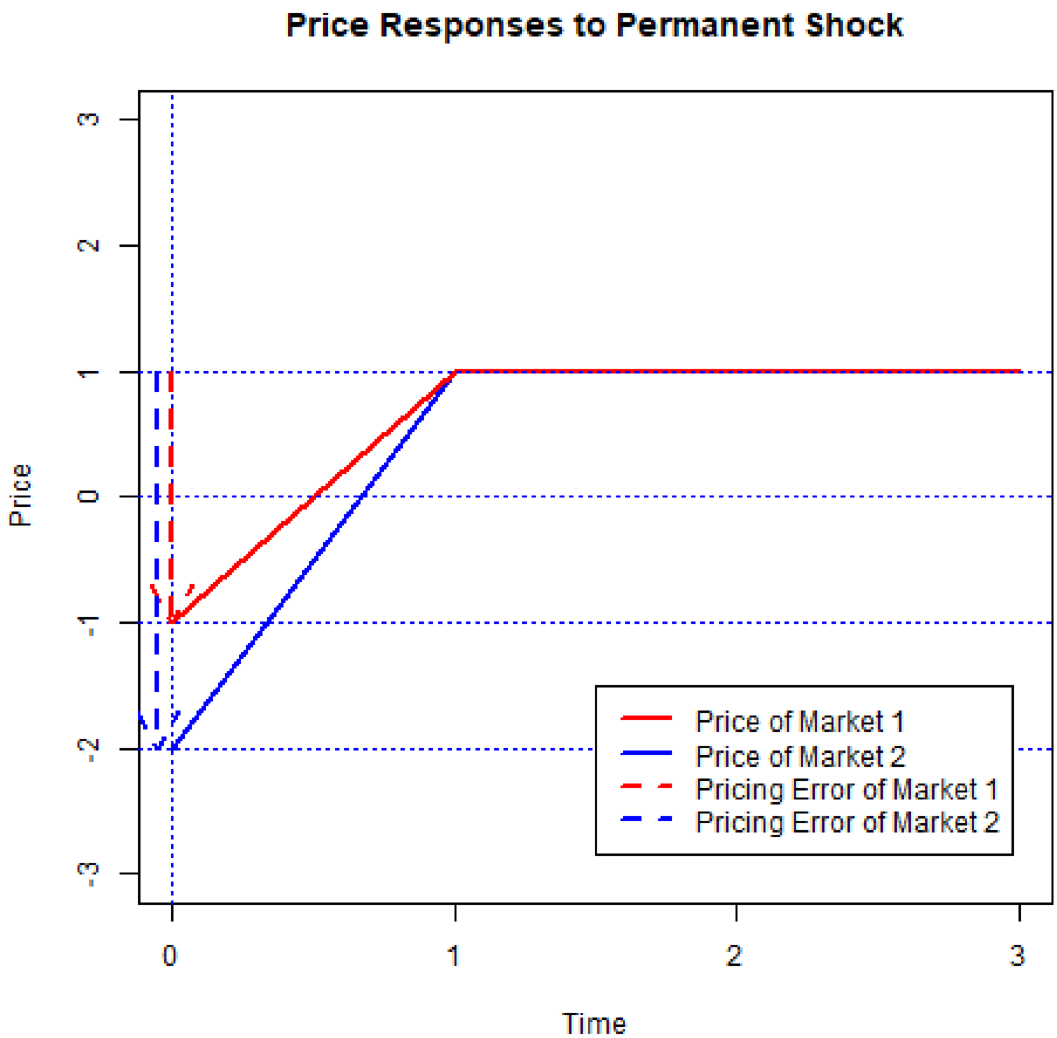

Figure 1: Price Responses to Permanent Shock

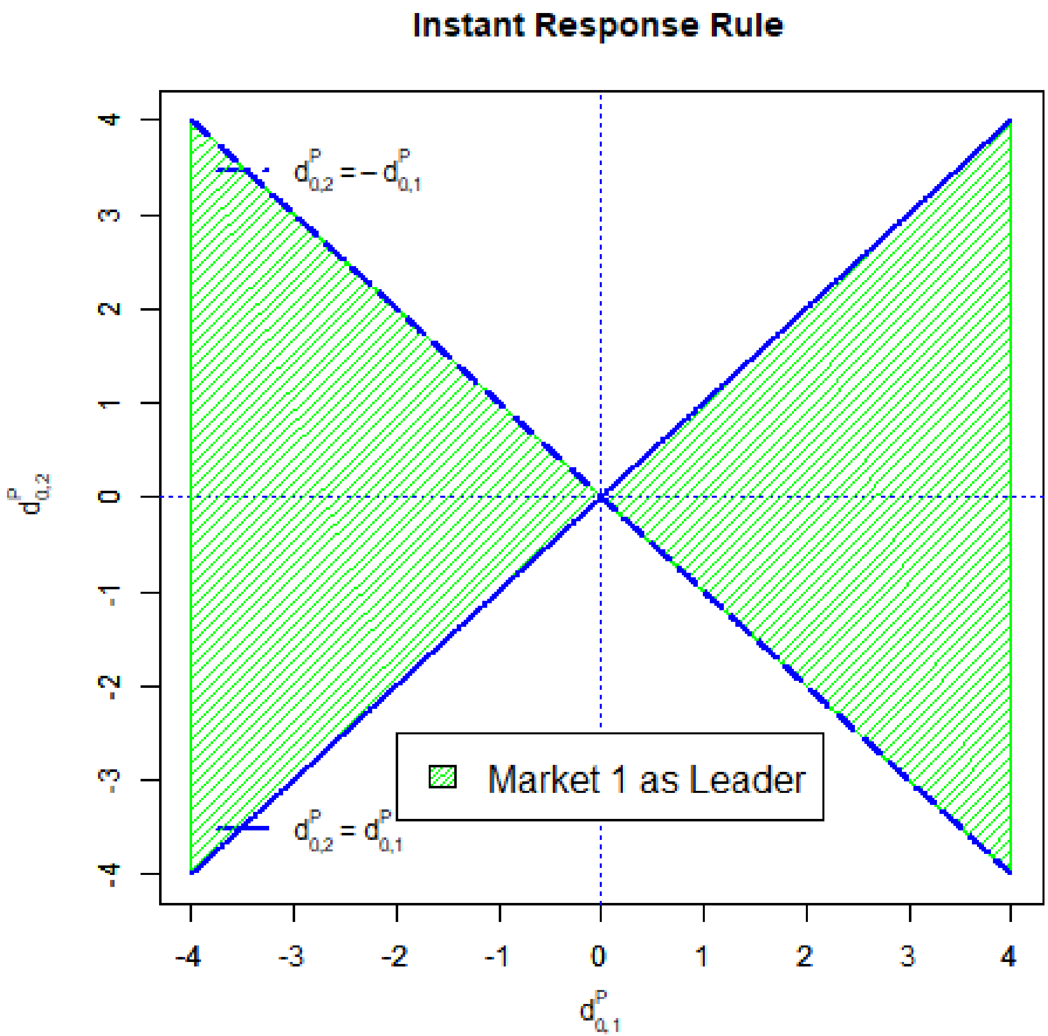

Figure 2: Instant Response Rule

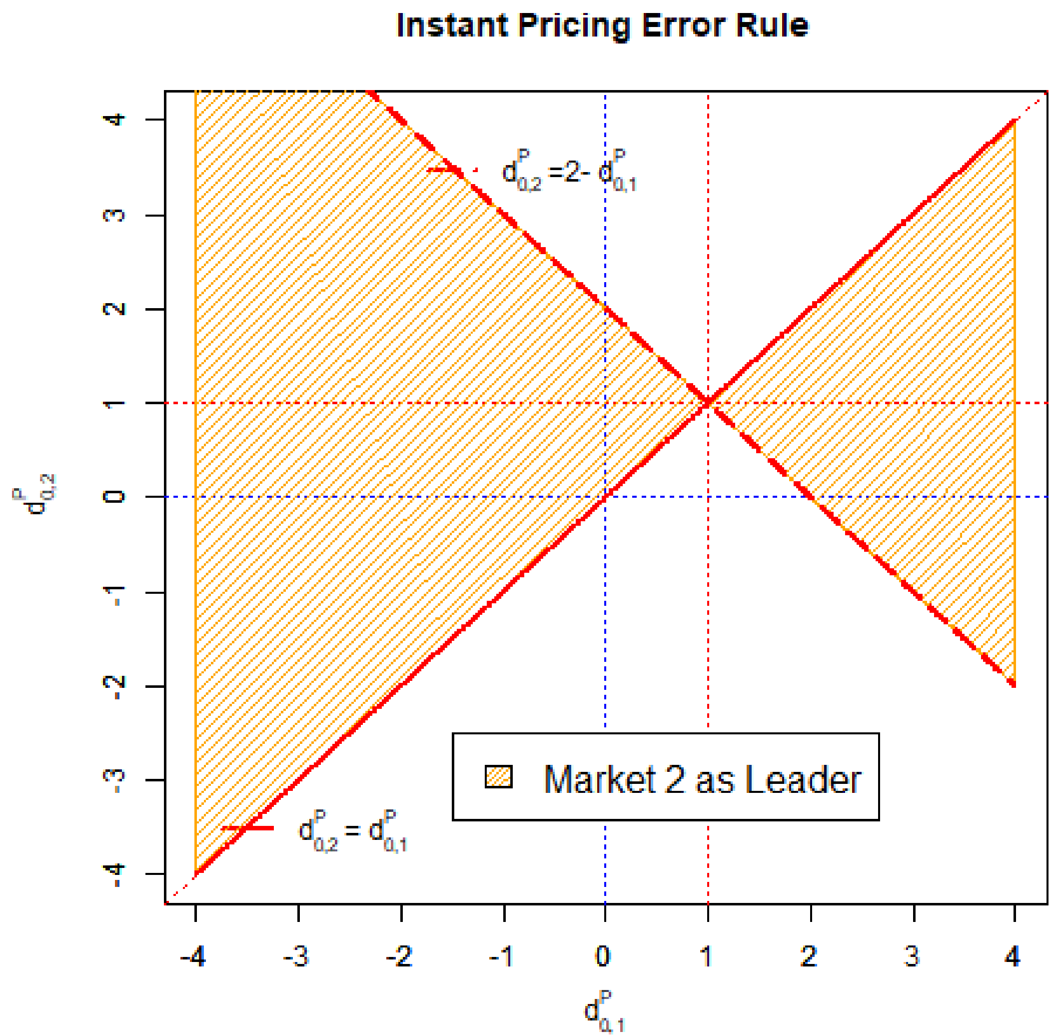

Figure 3: Instant Pricing Error Rule

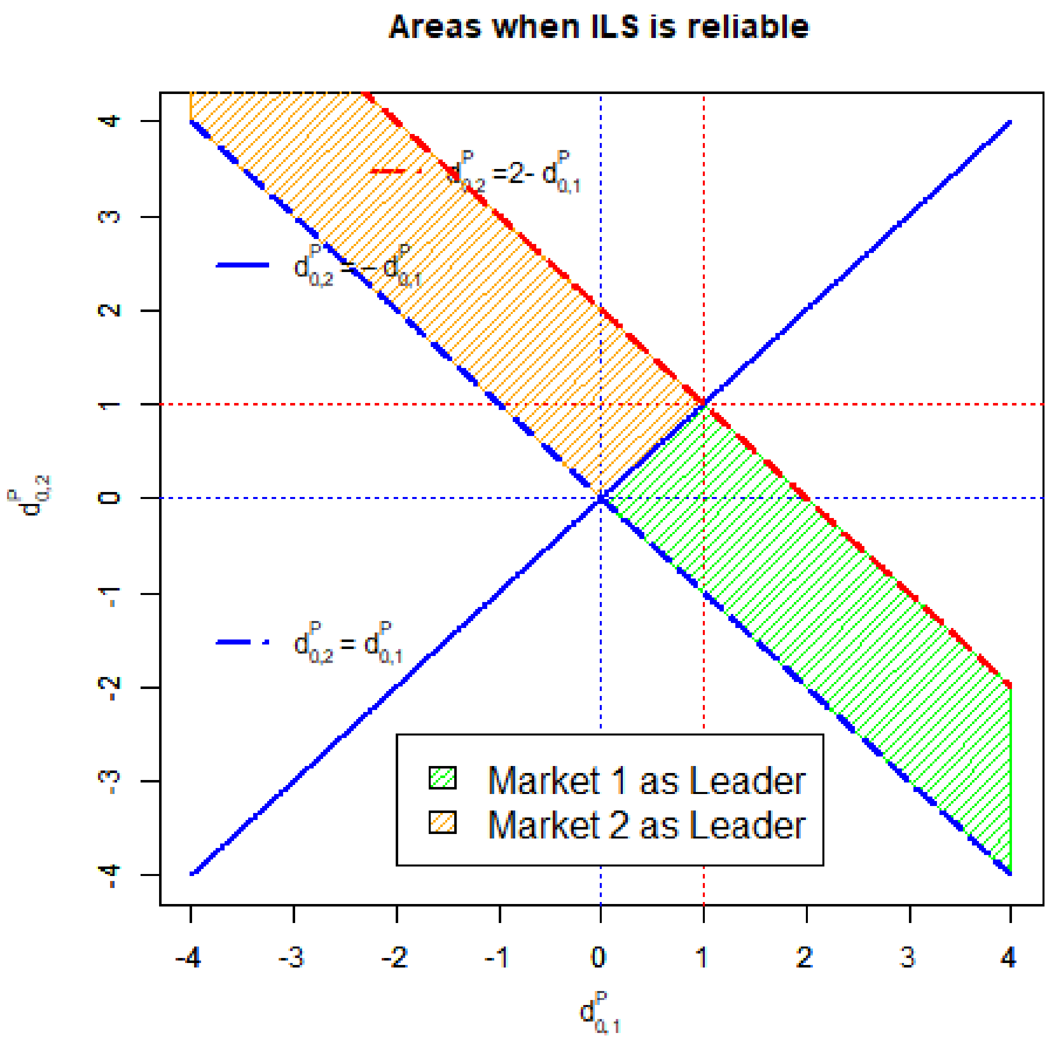

Figure 4: Areas when ILS is reliable